BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume VI Numéro 9

VARSOVIE 1958

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique.

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie.

Adresse de la Rédaction: Pałac Kultury i Nauki, Varsovie

BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

VOLUME VI
NUMÉRO 9

VARSOVIE 1958

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, WARSZAWA, MIODOWA 10

Nakład 850 + 120 egz.	*Rękopis dostarczono 4. VII. 1958*
Ark. wyd. 3,75 ark. druk. 4,25	*Podpisano do druku 27. XI. 1958*
Papier druk. sat. 80 g kl. III	*Druk ukończono w grudniu 1958*
Format B5, 70×100 cm	*Zam. prod. 483/58 Cena zł 20.–*

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHÉMATIQUE

Les solutions du système d'équations $\varphi(x) = \varphi(y)$ et $\sigma(x) = \sigma(y)$ pour $x<y<10000$

par

S. JANKOWSKA

Présenté par W. SIERPIŃSKI le 6 juin 1958

À l'aide des tables de J. W. L. Glaisher "Number-Divisor Tables" j'ai trouvé toutes les solutions du système d'équations

$$(1) \qquad \begin{cases} \varphi(x) = \varphi(y)\,, \\ \sigma(x) = \sigma(y) \end{cases}$$

en nombres naturels x et y, où $x < y < 10000$. Il existent 86 telles solutions que je donne dans la table I.

Parmi ces solutions il n'y a que cinq, où les nombres x et y sont premiers entre eux: ce sont les solutions

$$\begin{array}{cccccc} x & 175 & 496 & 2108 & 2896 & 7595\,, \\ y & 183 & 525 & 2145 & 2925 & 7881\,. \end{array}$$

De chaque solution en nombres naturels x et y du système d'équations (1) on peut sans peine obtenir une infinité d'autres, en remplaçant les nombres x et y par les nombres ax et ay, où a est un nombre naturel quelconque premier avec xy.

D'autre part, si x, y est une solution du système d'équations (1) et s'il existe un nombre $a > 1$ tel que $a|x$, $a|y$ et qu'en posant $x = ax_1$, $y = ay_1$, on a $(a, x_1y_1) = 1$, les nombres x_1 et y_1 présentent une solution des équations (1) en nombres naturels respectivement plus petits que x et y. Si un tel nombre a n'existe pas, nous dirons que la solution x, y est primitive.

TABLE I

x	y	$\varphi(x)$	$\delta(x)$	x	y	$\varphi(x)$	$\delta(x)$
175	183	120	248	4960	5236	1920	12096
220	246	80	504	4968	5970	1584	14400
350	366	120	744	5060	5658	1760	12096
352	410	160	756	5075	5307	3360	7440
496	525	240	992	5112	5742	1680	14040
568	638	280	1080	5166	5430	1440	13104
700	732	240	1736	5425	5673	3600	7936
888	970	384	1764	5456	5775	2400	11904
1056	1230	320	3024	5600	5856	1920	15624
1120	1326	384	3024	5740	6402	1920	14112
1400	1464	480	3720	5950	6222	1920	13392
1540	1722	480	4032	5984	6970	2560	13608
1704	1914	560	4320	6048	7410	1728	20160
1824	1836	576	5040	6102	6438	2016	13680
1925	2013	1200	2976	6188	6790	2304	14112
2108	2145	960	4032	6380	7134	2240	15120
2275	2379	1440	3472	6448	6825	2880	13888
2392	2492	1056	5040	6475	6771	4320	9424
2464	2870	960	6048	6650	6954	2160	14880
2583	2715	1440	4368	6688	7790	2880	15120
2652	2910	768	7056	6820	7626	2400	16128
2800	2928	960	7688	7175	7503	4800	10416
2840	3190	1120	6480	7176	7476	2112	20160
2860	3198	960	7056	7254	7602	2160	17472
2896	2925	1440	5642	7384	8294	3360	15120
2975	3111	1920	4464	7392	8610	1920	24192
3051	3219	2016	4560	7525	7869	5040	10912
3168	3690	960	9828	7595	7881	5040	10944
3325	3477	2160	4960	7700	8052	2400	20832
3627	3801	2160	5824	7749	8151	4320	13440
3640	3876	1152	10080	7956	8730	2304	22932
3740	4182	1280	9072	8050	8418	2640	17856
3850	4026	1200	8928	8096	9430	3520	18144
3944	4490	1792	8100	8140	9102	2880	19152
3976	4466	1680	8640	8225	8601	5520	11904
4025	4209	2640	5952	8370	8778	2160	23040
4180	4674	1440	10080	8432	8925	3840	17856
4185	4389	2160	7680	8520	9570	2240	25920
4484	4886	2088	8400	9100	9516	2880	24304
4550	4758	1440	10416	9120	9180	2204	30240
4576	5330	1920	10584	9184	9724	3840	21168
4720	5574	1856	11160	9275	9699	6240	13392
4940	5238	1728	11760	9424	9975	4320	19840

Pour $x < y < 10000$ il n'y a que 27 solutions primitives du système d'équations (1): elles sont données dans la table suivante:

x	y	x	y	x	y	x	y
175	183	1824	1836	3640	3876	4968	5970
220	246	2108	2145	3944	4490	5740	6402
352	410	2392	2492	4185	4389	6048	7410
496	525	2583	2715	4484	4886	7595	7881
568	638	2896	2925	4720	5574	7749	8151
884	970	3051	3219	4940	5238	9184	9724
1120	1326	3627	3801	4960	5236		

Je ne sais pas s'il existe une infinité de solutions primitives du système d'équations (1).

Parmi les solutions du système d'équations (1) il y en a de telles qui satisfont aussi à l'équation $\theta(x) = \theta(y)$, où $\theta(n)$ est le nombre de diviseurs naturels du nombre n. Pour $x < y < 10000$ ce sont les 16 solutions suivantes:

x	y	x	y	x	y	x	y
568	638	3051	3219	5112	5742	8370	8778
1704	1914	3976	4466	6102	6438	8520	9570
1824	1836	4185	4389	7384	8294	9120	9180
2840	3190	4960	5236	7749	8151	9184	9724

Il existe une infinité de systèmes de nombres naturels x et y, où $x < y$, satisfaisant à chacune de trois équations

$$\varphi(x) = \varphi(y), \quad \sigma(x) = \sigma(y), \quad \theta(x) = \theta(y);$$

tels sont, par exemple les nombres $x_k = 3^k \cdot 568$ et $y_k = 3^k \cdot 638$, où $k = 0, 1, 2, \ldots$

Parmi les systèmes de trois nombres naturels x, y, z, tels que $x < y < z < 10000$ il n'existe aucun tel que

$$\varphi(x) = \varphi(y) = \varphi(z) \quad \text{et} \quad \sigma(x) = \sigma(y) = \sigma(z).$$

Or, comme l'a remarqué W. Sierpiński, il existe une infinité de systèmes de trois nombres naturels x, y, z, où

$$x < y < z, \quad \varphi(x) = \varphi(y) = \varphi(z), \quad \sigma(x) = \sigma(y) = \sigma(z) \quad \text{et} \quad \theta(x) = \theta(y) = \theta(z).$$

Tels sont, par exemple, les systèmes x_k, y_k, z_k $(k = 0, 1, 2, \ldots)$, où $x_k = 5^k 2^3 3^3 71.113$, $y_k = 5^k 2^3 3.29.37.71$, $z_k = 5^k 2.3^3 11.29.113$.

Or, le problème se pose s'il existe pour tout nombre naturel m, m nombres naturels distincts $x_1, x_2, \ldots, x_m$, tels que

$$\varphi(x_1) = \varphi(x_2) = \ldots = \varphi(x_m), \quad \sigma(x_1) = \sigma(x_2) = \ldots = \sigma(x_m)$$
$$\text{et} \quad \theta(x_1) = \theta(x_2) = \ldots = \theta(x_m).$$

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHEMATICS

Solution of Two Problems of Jankowska

by

P. ERDÖS (Budapest)

Presented by W. SIERPIŃSKI on June 6, 1958

In the preceding paper Miss Jankowska puts the following two problems: I. Whether there exist infinitely many pairs of integers a and b satisfying $(a, b) = 1$, $\varphi(a) = \varphi(b)$, $\sigma(a) = \sigma(b)$, $d(a) = d(b)$, where $\varphi(n)$ is Euler's φ function, $\sigma(n)$ is the sum of divisors of n and $d(n)$ is the number of divisors of n. II. Whether for every k there exists a sequence of distinct integers $a_1, a_2, \ldots, a_k$ satisfying

$$\varphi(a_i) = \varphi(a_j), \quad \sigma(a_i) = \sigma(a_j) \quad \text{and} \quad d(a_i) = d(a_j)$$

for all $1 \leqslant i < j \leqslant k$.

Using the methods of one of my earlier papers [1] I am going to solve these problems and also state a few further problems.

First we need three lemmas:

LEMMA 1. *The number of integers not exceeding x all whose prime factors do not exceed $\log x$ is $o(x^{\varepsilon})$ for every $\varepsilon > 0$.*

LEMMA 2. *The number of squarefree integers not exceeding x composed of $c_1 \dfrac{(\log x)^{1+c_2}}{\log\log x}$ arbitrarily given primes not exceeding $(\log x)^{1+c_2}$ is greater than $c_3 x^{a}$, where a is any constant satisfying $0 < a < \dfrac{c_2}{2}$.*

Lemmas 1 and 2 are proved in [1] on pp. 211 and 212.

LEMMA 3. *We can find a constant c_2 so small that for a certain $c_1 > 0$ (in fact we only have to assume $c_1 < 1$) there are more than $c_1(\log x)^{1+c_2}$ primes p not exceeding $(\log x)^{1+c_2}$ such that both $p-1$ and $p+1$ are composed of primes not exceeding $\log x$.*

On p. 212-213 of [1] I proved an analogous lemma, where I required only that all prime factors of $p-1$ be less than $\log x$, but it is clear that the method used there (Brun's method) gives a proof of our Lemma 3.

Now we are ready to solve the problems of Miss Jankowska. Denote by $u_1 < u_2 < \ldots < u_l$ the squarefree integers composed of primes all whose

prime factors p do not exceed $(\log x)^{1+c_2}$ and such that all prime factors of $p+1$ and $p-1$ are less than $\log x$. By Lemmas 2 and 3 we obtain that, for sufficiently large x, $l > x^{c_2/4}$. On the other hand, all prime factors of $\varphi(u_i)$ and $\sigma(u_i)$, $1 \leqslant i \leqslant l$ are smaller than $\log x$. Thus, by Lemma 1, there are only $o(x^{\varepsilon})$ different values of $\varphi(u_i)$ and $\sigma(u_i)$ $1 \leqslant i \leqslant l$. The same holds for $d(u_i)$ since it is well known that $d(n) = o(n^{\varepsilon})$ for every $\varepsilon > 0$. Thus, there are $o(x^{3\varepsilon})$ choices for the triplet

$$\{\varphi(u_i), \sigma(u_i), d(u_i)\}, \quad 1 \leqslant i \leqslant l,$$

or there exist r integers $u_{i_1}, u_{i_2}, \ldots, u_{i_r}$ satisfying

$$r \geqslant \frac{l}{x^{3\varepsilon}} > x^{\frac{c_2}{4}-3\varepsilon}; \qquad \varphi(u_{i_1}) = \varphi(u_{i_2}) = \ldots = \varphi(u_{i_r}), \sigma(u_{i_1}) = \ldots = \sigma(u_{i_r});$$
$$d(u_{i_1}) = \ldots = d(u_{i_r}),$$

which completes the solution of the second problem of Jankowska.

It is clear that by the same method we can prove that for every r there exist k squarefree integers $a_1, a_2, \ldots, a_k$ satisfying

$$d(a_1) = d(a_2) = \ldots = d(a_k) \quad \text{and}$$
$$a_1 \prod_{p|a_1}\left(1+\frac{j}{p}\right) = a_2 \prod_{p|a_2}\left(1+\frac{j}{p}\right) = \ldots = a_k \prod_{p|a_k}\left(1+\frac{j}{p}\right)$$

for every $-r \leqslant j \leqslant r$, $j \neq 0$. The only change in the proof is that in Lemma 3 we have to require that all prime factors of $p+j$, $-r \leqslant j \leqslant r$, $j \neq 0$ be smaller than $\log x$.

To solve the first problem of Jankowska let a_i, b_i $1 \leqslant i \leqslant k$ satisfy

$$(1) \qquad (a_i, b_i) = 1, \quad \varphi(a_i) = \varphi(b_i), \quad \sigma(a_i) = \sigma(b_i).$$

Our proof will be complete if we succeed in finding another solution a_{k+1}, b_{k+1} of (1). But this is, indeed, easy. Let $v_1 < v_2 < \ldots < v_k \leqslant x$ be the squarefree integers composed of the primes p of Lemma 3, where we further require that $p \nmid \prod_{i=1}^{k} a_i b_i$. Since the last condition disqualifies only a bounded number of primes we obtain, by Lemma 2, that $k > x^{c_2/4}$ and we obtain, just as in the previous proof, two integers v_i and v_j satisfying

$$d(v_i) = d(v_j), \quad \varphi(v_i) = \varphi(v_j), \quad \sigma(v_i) = \sigma(v_j)$$

and no prime factor of $v_i v_j$ divides $\prod_{i=1}^{k} a_i b_i$. Put $(v_i, v_j) = t$. Then $a_{k+1} = \frac{v_i}{t}$, $b_{k+1} = \frac{v_j}{t}$ clearly satisfies (1), and thus the first conjecture of Jankowska is proved.

I conjecture that, for every k, there exists a sequence x_i, $1 \leqslant i \leqslant k$ of distinct integers satisfying

$$(x_i, x_j) = 1,\ 1 \leqslant i < j \leqslant k; \quad \varphi(x_1) = \ldots = \varphi(x_k),\ \sigma(x_1) = \ldots = \sigma(x_k); \quad d(x_1) = \ldots = d(x_k),$$

but I have not yet been able to prove this.

Denote by $A(n)$ the number of solutions of $\varphi(x) = n$. Heilbronn proved (in a letter to Davenport about 25 years ago) that

$$\frac{1}{x} \lim_{x\to\infty} \sum_{n=1}^{k} A^2(n) = \infty .$$

I believe that $\sum_{n=1}^{k} A(n)^2 > x^{2-\varepsilon}$. I have conjectured for a long time that for every $\varepsilon > 0$ and infinitely many n, $A(n) > n^{1-\varepsilon}$, but in [1] I could prove only that, for a certain $c > 0$ and infinitely many n, $A(n) > n^c$.

It is easy to see that if

$$(2) \qquad (x_i, x_j) = 1, \quad 1 \leqslant i < j \leqslant k \quad \text{and} \quad \varphi(x_1) = \varphi(x_2) = \ldots = \varphi(x_k) = n$$

then $k \leqslant d(n) < n^{c/\log\log n}$, since all prime factors of the x_i must be of the form $t+1$, $t|n$. On the other hand it can be deduced from results of Prachar [2] and myself that for infinitely many n we can have in (2) $k > n^{c/(\log\log n)^2}$.

Another problem would be to try to estimate the number of solutions in pairs of integers a and b of

$$(3) \qquad (a, b) = 1, \quad a < b < n, \quad \varphi(a) = \varphi(b).$$

It seems probable that the number of solutions is $> n^{2-\varepsilon}$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$.

Perhaps I may be permitted to mention the following problem of a different nature:

Can one find for every $\varepsilon > 0$ a sequence of consecutive integers $n+i$, $1 \leqslant i < n^{1-\varepsilon}$ satisfying $\varphi(n+i_1) \neq \varphi(n+i_2)$ for all $0 \leqslant i_1 < i_2 < n^{1-\varepsilon}$. I have not succeeded in solving this problem, not even with $\varepsilon > 1-\delta$ for any $\delta > 0$.

REFERENCES

[1] P. Erdös, Quarterly Journal of Math. **6** (1935), 205-213.
[2] K. Prachar, Monatshefte für Math. **59** (1955), 91-103.
[3] P. Erdös, Quarterly Journal of Math. **7** (1936), 227-229, see also S. Chowla, Proc. Indian Acad. Sci. Section A **5** (1937), 37-39.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHEMATICS

On the Summability of Bounded Sequences by Continuous Methods

by

W. ORLICZ

Presented on June 16, 1958

1. Let φ_n be continuous in the interval $\langle 0, T)$, where T may be finite or infinite. A sequence $x = \{t_n\}$ is termed *summable* to $\Phi(x)$ by the method Φ, corresponding to the sequence φ_n, if

(a) the series

$$(*) \qquad \Phi(\tau; x) = \sum_{\nu=1}^{\infty} \varphi_\nu(\tau) t_\nu$$

are convergent for $\tau \in \langle 0, T)$,

(b) there exists

$$\lim_{\tau \to T-} \Phi(\tau; x) = \Phi(x) .$$

The *field of summability* of the method Φ and the *zero-field* of summability of the method Φ (i. e. the set of all sequences summable by the method Φ and the set of all sequences summable to zero by the method Φ) will be denoted by Φ^* and Φ_0^*, respectively; the set of all sequences satisfying condition (a) and such that $\Phi(\tau; x)$ is a bounded function in $\langle 0, T)$, will be denoted by Φ^0. Moreover, the following notation will be applied: T = the space of all sequences, T_b = the space of bounded sequences, T_c = the space of convergent sequences, T_0 = the space of sequences convergent to zero, T_{f0} = the space of sequences having almost all terms equal to zero; in considering all these spaces, the usual definitions of addition of elements and multiplication of an element by a number and the usual definitions of the norm will be applied.

1.1. A method Φ is termed *continuous*, if the series $(*)$ is uniformly convergent in every interval $\langle 0, t\rangle$, where $0 < t < T$, for any sequence satisfying condition (a). A continuous method Φ is convergence-pre-

serving, i. e. $T_c \subset \Phi^*$, if, and only if, the following three conditions are satisfied:

$$1^\circ \quad L = \sup_{\tau} \sum_{\nu=1}^{\infty} |\varphi_\nu(\tau)| < \infty ;$$

$$2^\circ \quad \lim_{t\to T-} \sum_{\nu=1}^{\infty} \varphi_\nu(t) = a ;$$

$$3^\circ \quad \lim_{t\to T-} \varphi_n(t) = a_n \quad \text{for} \quad n = 1, 2, \ldots$$

If conditions $1^\circ - 3^\circ$ are satisfied and if the sequence $x = \{t_n\}$ is convergent to the limit γ, then

$$\Phi(x) = \sum_{\nu=1}^{\infty} a_\nu t_\nu + \gamma\chi(\Phi), \qquad \text{where} \qquad \chi(\Phi) = a - \sum_{\nu=1}^{\infty} a_\nu .$$

A necessary and sufficient condition for the permanence (permanence for null sequences) of a continuous method Φ is that conditions $1^\circ - 3^\circ$ and the condition $a = 1$, $a_n = 0$ for $n = 1, 2, \ldots$ (conditions 1°, 2° and the condition $a_n = 0$ for $n = 1, 2, \ldots$) be satisfied.

It will be noted that Włodarski gives in [6] a similar, but somewhat less general, definition of a continuous method. The proofs of the above theorems do not differ from those of the analogous theorems in [6].

1.2. *Let a continuous method Φ be convergence-preserving; then the series*

$$\sum_{\nu=1}^{\infty} |\varphi_\nu(\tau)|$$

is uniformly convergent in every interval $\langle 0, t)$, where $0 < t < T$.

Condition 1° implies convergence of the series $\sum \eta_\nu \varphi_\nu(t)$ in $\langle 0, T)$ for any zero-one sequence $\{\eta_n\}$; hence, it is uniformly convergent in $\langle 0, t)$, $t < T$, and, consequently, the series $\sum |\varphi_\nu(t)|$ is uniformly convergent in $\langle 0, t)$, $t < T$ (cf [4]).

It is the aim of this paper to give certain necessary and sufficient conditions for the field of summability of a continuous and convergence-preserving method to present a peculiar structure; namely, the field should contain no bounded divergent sequence. We are interested not only in formulating conditions for continuous methods, analogous to those of [1], [5] for matrix methods, but also in proving these theorems solely on the basis of typical functional-analytic methods, without using special lemmas as those applied in the papers referred to above.

2. *Let f_n denote continuous functions in $\langle 0, T\rangle$ satisfying the following conditions:*

$$\text{(a)} \quad \sup_{\langle 0,T)} \sum_{\nu=1}^{\infty} |f_\nu(t)| < \infty ,$$

(b) *the series*

$$\sum_{\nu=1}^{\infty} |f_\nu(t)|$$

is uniformly convergent in every interval $\langle 0, t\rangle$, *where* $t < T$.

If the inequality

$$(*) \qquad \sup_{\langle 0,T)} \Big|\sum_{\nu=1}^{\infty} \lambda_\nu f_\nu(t)\Big| \geqslant c \sup |\lambda_n|$$

holds for any sequence $\{\lambda_n\} \in T_{f0}$, c *being a positive constant, then the inequality*

$$(**) \qquad \max\Big[\sup_{\langle 0,T)} \Big|\sum_{\nu=1}^{\infty} \lambda_\nu f_\nu(t)\Big|, \ \Big|\sum_{\nu=1}^{\infty} \lambda_\nu f_\nu(T)\Big|\Big] \geqslant c \sup |\lambda_n|$$

is satisfied for any sequence $\{\lambda_n\} \in T_b$. *Conversely,* $(**)$ *for any* $\{\lambda_n\} \in T_b$ *implies* $(*)$ *for any* $\{\lambda_n\} \in T_{f0}$.

Let $C\langle 0, T\rangle$ denote the B-space of continuous functions in $\langle 0,T\rangle$ with the usual norm $\|x\|_c = \sup\limits_{\langle 0,T\rangle} |x(t)|$ and let b_n be real numbers, satisfying the condition $\sum\limits_{\nu} |b_\nu| < \infty$. The functions $x_n = f_n(t)$ belong to $C\langle 0, T\rangle$; hence, $(*)$ implies the inequality

$$\sum_{\nu=1}^{\infty} |b_\nu| \cdot \Big\|\sum_{\nu=1}^{p} \lambda_\nu x_\nu\Big\|_c \geqslant c \Big|\sum_{\nu=1}^{p} b_\nu \lambda_\nu\Big|$$

for arbitrary $\lambda_1, \lambda_2, \ldots, \lambda_p$. Now, by the theorem on moments and by the theorem of F. Riesz on the general form of a linear functional ξ over $C\langle 0, T\rangle$, a function $g(t)$ of bounded variation in $\langle 0, T\rangle$ exists with the following properties: $g(0) = 0$,

$$\|\xi\| = \underset{\langle 0,T\rangle}{\operatorname{var}}\, g(t) \leqslant \frac{1}{c} \sum_{\nu=1}^{\infty} |b_\nu|,$$

$$\int_0^T f_n(t)\, dg(t) = b_n \qquad \text{for} \qquad n = 1, 2, \ldots\ {}^{*)}.$$

---

*) If $T = \infty$, $\int\limits_0^T f_n(t)\, dg(t)$ is to mean as $\int\limits_0^{T-} f_n(t)\, dg(t) + f_n(T)[g(T) - g(T-)]$.

Choose a sequence $\tau_i \uparrow T$, $\tau_0 = 0$; then

$$b_n = \int\limits_0^T f_n(t)\,dg(t) = \sum_{i=1}^{\infty} \int\limits_{\tau_{i-1}}^{\tau_i} f_n(t)\,dg(t) + f_n(T)[g(T) - g(T-)],$$

$$\Big|\sum_{\nu=1}^{\infty} \pm \int\limits_{\tau_{i-1}}^{\tau_i} f_\nu(t)\,dg(t)\Big| = \Big|\int\limits_{\tau_{i-1}}^{\tau_i} \Big[\sum_{\nu=1}^{\infty} \pm f_\nu(t)\Big] dg(t)\Big| \leqslant L \operatorname*{var}_{\langle\tau_{i-1},\tau_i\rangle} g(t)$$

and it follows that

$$\sum_{\nu,i=1}^{\infty} \Big|\int\limits_{\tau_{i-1}}^{\tau_i} f_\nu(t)\,dg(t)\Big| \leqslant L \operatorname*{var}_{\langle 0,T\rangle} g(t).$$

Let λ_n be an arbitrary bounded sequence. The last inequality shows that the order of summation may be changed; hence,

$$\sum_{n=1}^{\infty} \lambda_n b_n = \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{i=1}^{\infty} \int\limits_{\tau_{i-1}}^{\tau_i} f_n(t)\,dg(t)\Big) + \Big(\sum_{n=1}^{\infty} \lambda_n f_n(T)\Big)[g(T) - g(T-)]$$

$$= \sum_{i=1}^{\infty} \int\limits_{\tau_{i-1}}^{\tau_i} \Big(\sum_{n=1}^{\infty} \lambda_n f_n(t)\Big) dg(t) + \Big(\sum_{n=1}^{\infty} \lambda_n f_n(T)\Big)[g(T) - g(T-)]$$

$$\leqslant \sup_{\langle 0,T)} \Big|\sum_{n=1}^{\infty} \lambda_n f_n(t)\Big| \Big(\sum_{i=1}^{\infty} \operatorname*{var}_{\langle\tau_{i-1},\tau_i\rangle} g(t)\Big) + \Big(\sum_{n=1}^{\infty} \lambda_n f_n(T)\Big)[g(T) - g(T-)]$$

$$= \sup_{\langle 0,T)} \Big|\sum_{n=1}^{\infty} \lambda_n f_n(t)\Big| \big[\operatorname*{var}_{\langle 0,T\rangle} g(t) - |g(T) - g(T-)|\big] +$$

$$+ \Big(\sum_{n=1}^{\infty} \lambda_n f_n(T)\Big)[g(T) - g(T-)]$$

and

$$\sum_{n=1}^{\infty} \lambda_n b_n \leqslant \max\Big[\sup_{\langle 0,T)} \Big|\sum_{n=1}^{\infty} \lambda_n f_n(t)\Big|,\ \Big|\sum_{n=1}^{\infty} \lambda_n f_n(T)\Big|\Big] \frac{1}{c} \sum_{n=1}^{\infty} |b_n|.$$

Choosing $b_n = \operatorname{sign}\lambda_n$, $b_i = 0$ for $i \neq n$ in the last inequality yields (**).

To prove that (**) implies (*), it is sufficient to note that, for sequences having almost all terms equal to zero, the inequality

$$\sup_{\langle 0,T)} \Big|\sum_{\nu=1}^{\infty} \lambda_\nu f_\nu(t)\Big| \geqslant \Big|\sum_{\nu=1}^{\infty} \lambda_\nu f_\nu(T)\Big|$$

holds.

2.1. Let $\|x\| = \sup|t_n|$; T_b, T_0 and T_c are Banach spaces with respect to this norm. Let Φ denote a continuous method of summability corresponding to a sequence φ_n. In Φ^0 the following pseudonorms will be defined:

(a) $\|x\|_0 = \sup\limits_{\langle 0,T)} |\Phi(\tau; x)|$,

(b) $\|x\|_{2i-1} = \sup\limits_{\langle 0,\tau_i\rangle} \sup\limits_{m} \left|\sum\limits_{\nu=1}^{m} \varphi_\nu(\tau) b_\nu\right|$, τ_i being fixed numbers, $\tau_i \uparrow T$,

(c) $\|x\|_{2i} = |t_i|$.

It can be easily proved that Φ^0 is a B_0-space with respect to these pseudonorms; this space will be denoted by $[\Phi^0; \| \|_i]$. Obviously, the co-ordinates are linear functionals over $[\Phi^0; \| \|_i]$. Φ^* is a linear closed subspace of $[\Phi^0; \| \|_i]$, to be denoted by $[\Phi^*; \| \|_i]$.

2.2. THEOREM. *Let Φ be a continuous convergence-preserving method; then the following conditions are mutually equivalent:*

(a) *the set T_c is closed in $[\Phi^*; \| \|_i]$;*

(a′) *the set T_0 is closed in $[\Phi^*; \| \|_i]$;*

(a″) *the set $\Phi^* \cap T_b$ is closed in $[\Phi^*; \| \|_i]$;*

(b) *the set T_b is closed in $[\Phi^0; \| \|_i]$;*

(c) *a positive integer r and a constant $c > 0$ exist such that the inequality*

$$\sup_{\langle 0,T)} \left|\sum_{\nu=1}^{p} \varphi_{\nu+r}(\tau)\lambda_\nu\right| \geqslant c \sup|\lambda_n|$$

holds for an arbitrary system of numbers $\lambda_1, \lambda_2, \ldots, \lambda_p$;

(d) *a positive integer r' and a constant $c' > 0$ exist such that the inequality*

$$\sup_{\langle 0,T)} \left|\sum_{\nu=1}^{\infty} \varphi_{\nu+r'}(\tau)\lambda_\nu\right| \geqslant c' \sup|\lambda_n|$$

holds for any bounded sequence λ_n;

(e) *Φ^* contains no bounded divergent sequence.*

First, the implication (a)$\to$(c) will be proved.

The functionals $\xi_n(x) = t_n$ are linear over T_c (considered as a subspace of $[\Phi^*; \| \|_i]$) and $\xi_n(x) = O(1)$ in every point of the space T_c; hence, by a well-known theorem, a constant $k > 0$ and a positive integer s exist such that the inequality

$$|\xi_n(x)| \leqslant k \max_{i \leqslant s}(\|x\|_i) \quad \text{for} \quad x \in T_c$$

holds. Here, the number s may be assumed even. According to 1.2 there exists a positive integer $r \geqslant s/2$ such that

$$\sum_{\nu=r+1}^{\infty} |\varphi_\nu(t)| \leqslant \frac{1}{2k} \quad \text{for} \quad t \in \langle 0, \tau_{s/2}\rangle.$$

Considering an arbitrary convergent sequence $\{t_n\}$ with the first r terms equal to zero, the following inequalities will be obtained:

$$\sup_n |t_n| \leqslant k \max_{1\leqslant i\leqslant s} (\|x\|_i) \leqslant k\left(\sup |t_n|\cdot\frac{1}{2k}+\sup_{\langle 0,T)}|\Phi(\tau; x|\right);$$

hence, inequality (c) results. The implications (a$'$)$\to$(c), (a$''$)$\to$(c) and (b)$\to$(c) may be proved similarly.

In order to prove the implication (d)$\to$(a) let

$$x_k = \{t_n^k\} \epsilon T_c, \qquad x = \{t_n\} \epsilon \Phi^* \quad \text{and} \quad \|x_k - x\|_i \to 0$$

as $k\to\infty$ for $i = 1, 2, \ldots$ It follows from the inequality (d) that

$$\sup_{\langle 0,T\rangle}\left|\sum_{\nu=1}^{\infty}\varphi_{\nu+r'}(\tau)(t^p_{\nu+r'}-t^q_{\nu+r'})\right| \geqslant c' \sup_{n>r'}|t_n^p - t_n^q|.$$

Since $\|x_p - x_q\|_i \to 0$ as $p, q\to\infty$ for $i = 1, 2, \ldots$, the left side of the above inequality tends to 0; hence, $\sup\limits_n |t_n^p - t_n^q| \to 0$, and the relation $t_n^k \to t_n$ for $n = 1, 2, \ldots$ implies $\sup\limits_n |t_n^k - t_n| \to 0$. Thus, $\{t_n\}\epsilon T_c$. The implications (d)$\to$(a$'$), (d)$\to$(a$''$) and (d)$\to$(b) may be proved similarly.

Now, the proof of (c)$\to$(d) will be given. According to 2 and (c), the inequalities

$$\sup_{\langle 0,T)}\left|\sum_{\nu=1}^{\infty}\varphi_{\nu+r}(\tau)\lambda_\nu\right| + \left|\sum_{\nu=1}^{\infty}a_{\nu+r}\lambda_\nu\right|$$
$$\geqslant \max\left[\sup_{\langle 0,T)}\left|\sum_{\nu=1}^{\infty}\varphi_{\nu+r}(\tau)\lambda_\nu\right|, \left|\sum_{\nu=1}^{\infty}a_{\nu+r}\lambda_\nu\right|\right] \geqslant c \sup|\lambda_n|$$

are valid for any bounded sequence λ_n. Choosing $r'\geqslant r$ such that $\sum\limits_{\nu=1}^{\infty}|a_{\nu+r'}| < c/2$, $\lambda_{r+1} = \ldots = \lambda_{r'} = 0$, (d) is obtained. The converse implication (d)$\to$(c) is obvious.

To prove (a$'$)$\to$(e), let Φ denote the method corresponding to the sequence $\bar\varphi_n(t) = \varphi_n(t) - a_n$ for $n = 1, 2, \ldots$, which satisfies the conditions 1.1. 1°-3°, $a_n = 0$. Choose a positive integer s such that $\sum\limits_{\nu>s}|a_\nu| < c/2$, $s\geqslant r$; then (c) implies

$$(*)\quad \sup_{\langle 0,T)}\left|\sum_{\nu=1}^{\infty}\bar\varphi_{\nu+s}(\tau)t_{\nu+s}\right| \geqslant \sup_{\langle 0,T)}\left|\sum_{\nu=1}^{\infty}\varphi_{\nu+s}(\tau)t_{\nu+s}\right| - \left|\sum_{\nu=1}^{\infty}t_{\nu+s}a_{\nu+s}\right|$$
$$\geqslant \frac{c}{2}\sup_{n>s}|t_n| \quad \text{for every} \quad x\epsilon T_{f0}.$$

By a well-known lemma ([7], p. 192), for any bounded sequence x $\bar{\Phi}$-summable to 0 and for any $\varepsilon > 0$ and a positive integer k, a sequence $y \in T_{f0}$ exists with the following property: the first k terms of the sequence x are equal to the first k terms of the sequence y, respectively, and $\sup\limits_{\langle 0,T)} |\bar{\Phi}(\tau; x-y)| < \varepsilon$. Hence, it results that the inequality (*) is also satisfied for $x \in \bar{\Phi}_0^* \frown T_b$ (obviously, (*) results from (d) also, though with another constant). Choosing $k > s$, (*) implies

$$\varepsilon > \sup_{\langle 0,T)} |\bar{\Phi}(\tau; x-y)| \geqslant \frac{c}{2} \sup_{n>s} |t_n - t_n^0|;$$

hence, $\bar{\Phi}_0^* \frown T_b \subset T_0$. Further, we have $\chi(\Phi) \neq 0$; indeed, $\chi(\Phi) = 0$ would imply that the sequence $e = 1, 1, \ldots, 1, \ldots$ is $\bar{\Phi}$-summable to 0 and that $e \in T_0$, which is impossible. Let

$$a = \frac{\Phi(x) - \sum\limits_{\nu=1}^{\infty} a_\nu t_\nu}{\chi(\Phi)}, \quad \text{wherein} \quad x \in \Phi_b^*.$$

It is easily seen that now the sequence $\{t_n - a\} \in \bar{\Phi}_0^* \frown T_b$, whence $\{t_n - a\} \in T_0$ and $\{t_n\} \in T_c$.

Finally, we give the proof of the implication (e)→(c). Let $\bar{\Phi}$ be the same method of summability as above; $\Phi^* \frown T_b = \bar{\Phi}^* \frown T_b$. In $\bar{\Phi}_0^* \frown T_b \subset T_c$ the norms will be defined by the formulae

$$\|x\| = \sup |t_n|, \quad \|x\|^* = \sum_{n=1}^{\infty} |t_n|/2^n + \sup_{\langle 0,T)} |\bar{\Phi}(\tau; x)|.$$

By these norms, the set $X_s = \{x : x \in \bar{\Phi}_0^* \frown T_b, \|x\| \leqslant 1\}$ is a Saks space satisfying the condition (Σ_1) (see [2], [3], especially p. 23). If the φ_n are replaced by $\bar{\varphi}_n$, then the inequality (c) is satisfied with a positive integer $\bar{r}$ and a constant $\bar{c} > 0$. Indeed, in the contrary case, there would exist sequences $x^k = \{t_n^k\} \in T_{f0}$ such that $t_n^k = 0$ for $1 \leqslant n \leqslant k$, where $k = 1, 2, \ldots$, $\sup |t_n^k| = 1$ for $k = 1, 2, \ldots$ and $\sup\limits_{\langle 0,T)} |\bar{\Phi}(\tau; x^k)| \to 0$ as $k \to \infty$. Since $T_{f0} \subset \bar{\Phi}_0^* \frown T_b$, there follows $x^k \in X_s$ and $\|x^k\|^* \to 0$. On the other hand, $\xi_n(x) = t_n$ being linear functionals over X_s, convergent everywhere and X_s satisfying the condition (Σ_1), a well-known theorem ([3], theorem 7) implies ξ_n to be equicontinuous at 0. Hence, $\sup |\xi_n(x^k)| = \sup |t_n^k| \to 0$ and this results in a contradiction. If the inequality (c) is valid for $\bar{\varphi}_n$, then it is valid with a suitable $r \geqslant \bar{r}$ and with the constant $c = \bar{c}/2$ for the functions φ_n, too.

The above theorem will be completed by the following remark. Let us assume a continuous, convergence-preserving method Φ satisfying the equality $\Phi^* \frown T_b = \Phi^*$. Then condition (a″) is satisfied; hence, (a″) is

satisfied, too. Now, (a′) implies $\chi(\Phi) \neq 0$, a fact demonstrated in the course of the proof of the implication (a′)→(e). Moreover, (e) must be satisfied. Hence, we obtain the following

COROLLARY: *If Φ is a continuous, convergence-preserving method and if one of the following conditions holds:*

(α) $\chi(\Phi) = 0$,

(β) *there exists in Φ^* a bounded, divergent sequence,*

then an unbounded, divergent, Φ-summable sequence exists (see [7], [3]).

From the equivalence-theorem presented above, an analogous theorem for matrix methods may be obtained by a suitable specialization of $\varphi_n(t)$. Let A denote the matrix method corresponding to the matrix (a_{in}). Then a continuous method Φ corresponding to the functions $\varphi_n(t)$ will be defined by the formulae $\varphi_i(n) = a_{in}$ for $n = 1, 2, \ldots$, $\varphi_i(t) =$ a linear function in each of the intervals $n-1 \leqslant t \leqslant n$, $\varphi_i(0) = 0$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] S. Mazur, W. Orlicz, *On linear methods of summability*, Studia Mathem. **14** (1954), 129-160.

[2] W. Orlicz, *Linear operations in Saks spaces (I)*, Studia Mathem. **11** (1950), 237-272.

[3] — *Linear operations in Saks spaces (II)*, Studia Mathem. **15** (1955), 1-25.

[4] — *O szeregach doskonale zbieżnych...*, Prace Matem. **1** (1954), 393-414.

[5] A. Wilansky, K. Zeller, *Summation of bounded divergent sequences, Topological methods*, Transaction Am. Math. Soc. **78** (1955), 501-509.

[6] L. Włodarski, *Sur les méthodes continues de limitation (I)*, Studia Mathem. **14** (1954), 161-187.

[7] — *Sur les méthodes continues de limitation (II)*, Studia Mathem. **14** (1954), 188-199.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHEMATICS

A Symmetrization Result for Maximum Modulus

by

J. KRZYŻ

Presented by A. MOSTOWSKI on June 23, 1958

Let us consider a class of functions $f(z)$ regular in the unit circle and such that Riemann surfaces being maps of the unit circle under $f(z)$ fulfil certain geometrical conditions. The conditions imposed which define the class, may be of different kinds, e.g. omitting of a given value and univalency, or p-valency in the ordinary sense or some generalized sense, say in the sense of Biernacki [1], or Hayman [3]. It is a rather general fact that the extremal function with the greatest maximum modulus for each class represents the unit circle on a Riemann surface with circular symmetry. This means that the Riemann surface corresponding to the extremal function is invariant under circular symmetrization (for notion and properties of circular symmetrization see [6], [7].

According to the fundamental theorem of Riemann and its generalizations due to Poincaré and Koebe, any simply connected Riemann surface (over the w-plane) of hyperbolic type with a distinguished linear element on it may be considered as a function $f(z)$, regular in $|z| < 1$, inasmuch as $f(z)$ realizes the biuniform conformal representation of $|z| < 1$ onto the given Riemann surface W, and a distinguished direction at $z = 0$ and the given linear element at W correspond to each other.

In this paper we shall prove that the function $f^*(z)$ corresponding to the Riemann surface W^*, obtained from W by circular symmetrization, increases more rapidly than $f(z)$, i. e. the function corresponding to W. We can assume without loss of generality that W is bounded and that its boundary is an analytic curve. The general case can be obtained from this by using the Carathéodory notion of the nucleus convergence for a sequence of Riemann surfaces [2] and a subsequent passage to limit. We must, however, assume in addition that W^* is also of a hyperbolic type which seemingly does not always occur.

THEOREM. *Let W be a simply connected (and bounded) Riemann surface over the w-plane being the biuniform map of the circle $|z| < 1$ by $f(z)$, $f(z)$ being regular in $|z| < 1 + \delta$ ($\delta > 0$) and such that $f(0) = 0$, $f'(0) > 0$,*

and $f(z)\neq 0$ for $0<|z|\leqslant 1$. Let W^* be a simply connected Riemann surface obtained by circular symmetrization of W with respect to the positive real axis and let $f^*(z)$ be the "symmetrized function" mapping the unit circle $|z|<1$ onto W^* and such that $f^*(0)=0$, $f^{*\prime}(0)>0$ (the latter condition being admissible since a neighbourhood of $w=0$ is simply covered by the points of W^*). If $w_0=f(z_0)\neq 0$, then there exists a positive z_0^* such that $|w_0|=f^*(z_0^*)$ and $z_0^*\leqslant|z_0|$. In other words, the "symmetrized function" $f^*(z)$ approaches earlier the value of a given modulus.

Proof. We slit the unit circle K on the z plane along a radius from 0 to z_0 and so obtain a doubly connected domain K_{z_0}. The line segment $0z_0$ is mapped by $f(z)$ onto a curve L lying on W and connecting the point $w=0$ with a point of W lying over w_0. The boundary of W and the curve L make up together a doubly connected Riemann surface W_L. By the conformal invariance of the modulus of a doubly connected Riemann domain we have

$$M(K_{z_0})=M(W_L),$$

the modulus $M(F)$ of a doubly connected Riemann surface F being defined as the reciprocal of the extremal length of the family of closed curves lying on the surface and separating both components of the boundary (for definition and properties of extremal length see [4]). Now, we symmetrize W_L (for the circular symmetrization of a doubly connected domain considered as a electrostatic condenser see [7], or [5]) and so obtain a doubly connected Riemann surface $W^*_{w_1}$ which is derived from W^* by rejecting the segment $0\,w_1$ over the positive real axis, where $w_1=\sup_{w\in L}|w|$. Since $w_0\in L$, we have $|w_0|\leqslant w_1$. The circular symmetrization, however, increases the modulus [7], [5] and therefore

$$M(W_L)\leqslant M(W^*_{w_1}).$$

The Riemann surface $W^*_{w_0}$, obtained from W^* by rejecting the segment $0w_0$ over the real axis, may be considered as originated from $W^*_{w_1}$ by its expansion. It is well known [7] that the modulus is a "monotonic" function — an expansion of the doubly connected domain increases it. Therefore,

$$M(W^*_{w_1})\leqslant M(W^*_{w_0}).$$

Now, by the symmetry, the map of the radius $0\leqslant z<1$ by $f^*(z)$ is the segment $0\leqslant w<M$ $(M=\sup_{|z|<1}|f(z)|=\sup_{|z|<1}|f^*(z)|)$ and therefore there exists a real positive z_0^*, $0<z_0^*<1$, such that $|w_0|=f^*(z_0^*)$.

Besides, by the conformal invariance of the modulus, we have

$$M(W^*_{w_0})=M(K_{z_0^*}).$$

The inequalities obtained imply $M(K_{z_0}) \leqslant M(K_{z_0^*})$ and this means that $z_0^* \leqslant |z_0|$ by the "expansion property" of the modulus. Our theorem is proved.

Taking now w_0 such that $M(r,f) = |w_0|$ $(M(r,f) = \sup_{|z|\leqslant r} |f(z)|)$, we see that $M(r,f) = f^*(r^*) \leqslant M(r^*, f^*)$, where $r^* \leqslant r$. This implies that

$$M(r,f) \leqslant M(r,f^*)\,. \tag{1}$$

The above obtained results are also valid for functions of the form: $f(z) = az^p + \ldots$ $(a > 0)$ and such that $f(z) \neq 0$ for $z \neq 0$, and W^* having a branch point of order p at $w = 0$. This can be reduced to the case just considered by taking the p-th root.

Inequality (1) is an obvious generalization of the well-known inequality $|f'(0)| \leqslant |f^{*\prime}(0)|$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] M. Biernacki, Bull. Sci. Math., **70** (1946), 51-76.
[2] C. Carathéodory, Math. Ann., **72** (1912), 107-144.
[3] W. K. Hayman, Journ. d'Analyse Math. **1** (1951), 155-179.
[4] J. A. Jenkins, Ann. of Math. Studies, **30**, (1953), 87-94.
[5] — Trans. of Amer. Math. Soc. **79** (1955), 423-427.
[6] G. Pólya, Comptes Rendus de l'Acad. Sci., **230** (1950), 25-27.
[7] G. Pólya and G. Szegö, Ann. of Math. Studies, **27** (1951),

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHEMATICS

A Simple Proof of an Imbedding Theorem of the Kondrashev Type

by

P. SZEPTYCKI

Presented by T. WAŻEWSKI on June 23, 1958

1. The aim of this paper is to give a short proof of the so-called Kondrashev imbedding theorem based on inequalities given by Nirenberg [1]. The original proof of this theorem contained in [3] is rather wearisome and complicated. Suppositions on the domain Ω_N in our formulation of the theorem are different to those required in [3], namely, we prove the theorem for an important class of domains defined by Nirenberg in [2].

2. Let Ω_N be a bounded domain of N-dimensional Euclidean space. We denote by $x, y, z, \ldots$ points of Ω_N, writing x instead of $(x_1, \ldots, x_N)$, by $D^\nu f$ — partial derivatives of order

$$|\nu| = \nu_1 + \ldots + \nu_N; \qquad D^\nu f = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \ldots \partial x_N^{\nu_N}},$$

writing $|D^l f|^p$ instead of the sum $\sum\limits_{|\nu|=l} |D^\nu f|^p$.

As usual $C(\Omega_N)$, $C^k(\Omega_N)$, $C^\infty(\Omega_N)$ denote respectively the spaces of all continuous, differentiable up to order k, or infinitely differentiable functions on $\Omega_N \cdot W_p^{(l)}(\Omega_N)$; $p > 1$ denotes the Sobolev space on Ω_N, i. e. the completion of $C^\infty(\Omega_N)$ in the sense of the norm

$$\|f\|_l^p = \sum_{m=0}^{l} |f|_m^p = \sum_{m=0}^{l} \int_{\Omega_N} |D^m f(x)|^p \, dx . \tag{1}$$

We say, that Ω_N possesses a strong cone property (Nirenberg [2]), if there exists such a fixed spherical segment Σ with a positive radius h and positive spherical angle that, for any two points $x, y \in \Omega_N$, there exist Σ_x, Σ_y isometric with Σ having centres in x and y, contained

in Ω_N. We also require that the volume of the set

$$\{z: z \in \sum_x \cap \sum_y; \quad |x-z|, |y-z| \leqslant |x-y|\}$$

be not less than $\lambda |x-y|^N$ for some positive λ independent of x and y.

3. We are going to prove the following

THEOREM (cf. Sobolev, [3]). *If Ω_N is any bounded domain in E_N possessing strong cone property, l is a natural number,*

$$l \geqslant \left[\frac{N}{P}\right]+1, \quad p > 1$$

then

$$W_p^{(l)}(\Omega_N) \subset C(\Omega_N)$$

and the imbedding

$$W_p^{(l)}(\Omega_N) \to C(\Omega_N)$$

is completely continuous.

The proof will follow immediately from the following lemmas.

LEMMA 1. *If $f \in C^\infty(\Sigma)$, Σ is a spherical segment with the centre 0, β being any number $0 < \beta < N$, then for $l \geqslant \left[\frac{\beta}{p}\right]+1$, $p > 1$*

$$(1') \qquad \int_\Sigma |x|^{-\beta} |f(x)|^p dx \leqslant C \|f\|_l^p \ (\text{on } \Sigma),$$

where the constant C depends only on Σ, β but does not depend on f.

Proof. Nirenberg proved (cf. [1]), that

$$(2) \qquad \int_{\frac{1}{2}\Sigma} |x|^{-\beta} |f(x)|^p dx \leqslant C\Big[\int_\Sigma |f(x)|^p dx + \int_\Sigma |x|^{-\beta+p} |Df(x)|^p dx\Big],$$

where $\frac{1}{2}\Sigma$ denotes the spherical segment $\{x: x = \frac{1}{2}y,\ y \in \Sigma\}$ and the constant C depends only on Σ and β. Consider the integral

$$(3) \qquad \int_{\Sigma-\frac{1}{2}\Sigma} |x|^{-\beta} |f(x)|^p dx = \int_{\Sigma'} \int_{h/2}^{h} r^{-\beta} |f|^p r^{N-1} dr\, d\omega \leqslant \left(\frac{h}{z}\right)^{-\beta} \int_\Sigma |f(x)|^p dx,$$

where r, ω denote spherical co-ordinates in Σ, h — its radius and Σ' — the domain of spherical angle ω.

By adding (2) and (3) we obtain

$$(4) \qquad \int_\Sigma |x|^{-\beta} |f(x)|^p dx \leqslant C\Big[\int_\Sigma |f(x)|^p dx + \int_\Sigma |x|^{-\beta+p} |Df(x)|^p dx\Big],$$

where C does not depend on f. Applying l-times the inequality (4) to its right side we obtain the assertion.

LEMMA 3 (Nirenberg). *Denote by Σ_x a spherical segment with the centre at x and radius h, let $p > 1$, and a, β be two positive numbers*

(5)
$$1 > a = 1 - \frac{N-\beta}{p} > 0 .$$

Then we have

(6)
$$\int_{\Sigma_x} |f(x) - f(z)| \leqslant Ch^{N+a} \Big(\int_{\Sigma_x} |z-x|^{-\beta} |Df(z)|^p dz \Big)^{1/p},$$

where C depends only on the spherical angle of Σ_x.

The simple proof of this lemma is given in [1].

Now, suppose Ω_N possesses the strong cone property and let, for any $x \in \Omega_N$, Σ_x be the spherical segment with centre in x isometric with Σ fixed for the whole Ω_N. For each $f \in C^\infty(\Omega_N)$ we have

(7)
$$|f(x)| \leqslant |f(x) - f(z)| + |f(z)|, \quad z \in \sum\nolimits_x .$$

Integrating (7) with respect to z over Σ_x, estimating the first term on the right side of the inequalities obtained by means of (4) and (1′), and the second one by means of Hölder's inequality we get

(8)
$$|f(x)| \varkappa h^N \leqslant C \sum_{m=0}^{[\beta/p]+2} \Big(\int_{\Sigma_x} |D^m f(z)|^p dz \Big)^{1/p} \leqslant C \|f\|^p_{[\frac{\beta}{p}+2]} ,$$

where C depends on Σ, p and β, while $\varkappa h^N$ is the volume of Σ_x. By the definition of $W_p^{(l)}(\Omega_N)$ we obtain from (8) the inclusion $W_p^{(l)}(\Omega_N) \subset C(\Omega_N)$ for $l \geqslant \beta/p + 2$ and the continuity of the imbedding. To prove its complete continuity consider two spherical segments Σ_x, Σ_y with centres in x and y satisfying conditions of the definition of the strong cone property. Denote

$$\sum\nolimits_x^* = \Big\{ z : z \in \sum\nolimits_x, \ |z-x| \leqslant |x-y| \Big\} \sum\nolimits_y^* = \Big\{ z : z \in \sum\nolimits_y, \ |z-y| \leqslant |x-y| \Big\} .$$

We have for each $f \in C^\infty(\Omega_N)$,

$$|f(x) - f(y)| \leqslant |f(x) - f(z)| + |f(y) - f(z)| .$$

Integrating this inequality with respect to z over $\Sigma_x^* \cap \Sigma_y^*$ we get

$$\lambda |x-y|^N |f(x) - f(y)| \leqslant \int_{\Sigma_x} |f(x) - f(z)| dz + \int_{\Sigma_y} |f(y) - f(z)| dz .$$

Applying (6) to the right side of the above inequality we obtain

(9)
$$|f(x) - f(y)| \leqslant C |x-y|^a \Big[\Big(\int_{\Sigma_x} |Df(z)|^p |x-z|^{-\beta} dz \Big)^{1/p} + \\ + \Big(\int_{\Sigma_y} |Df(z)|^p |y-z|^{-\beta} dz \Big)^{1/p} \Big].$$

Estimating the right side of (9) by means of (1′) we obtain the assertion that all functions contained in any fixed bounded set K in $W_p^{(l)}$, $l \geqslant [\beta/p]+2$ satisfy the Hölder condition with the same positive exponent a and with the same constant. At the same time these functions, according to (8), have a common bound and therefore the set K is compact in C. Note that we can choose β in such a way that

$$\left[\frac{\beta}{p}\right] = \left[\frac{N}{p}\right] - 1, \tag{9}$$

condition (5) being satisfied. Namely, we can put $\beta = N-p+\varepsilon$, $\varepsilon > 0$ satisfying the condition $[N+\varepsilon/p] = [N/p]$.

The last condition in the case $p = 2$ may be simplified, as follows: $a = \frac{1}{2}$ if N is even, $a < 1$ if N is odd.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] L. Nirenberg, Comm. on pure and appl. math. **9** (1956), 509-531.

[2] — ibid., **8** (1955), 648-674.

[3] S. L. Sobolev, *Some applications of functional analysis in mathematical physics* (in Russian), **13** (1958), 169-175.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHÉMATIQUE

Sur les composantes de l'espace des transformations d'un espace localement compact en un rétracte de voisinage

par

K. KURATOWSKI

Présenté le 30 Juin 1958

Dans cette Note, $\mathcal{X}$ et $\mathcal{Y}$ vont désigner deux espaces métriques séparables, dont $\mathcal{X}$ est un espace *localement compact* *) et $\mathcal{Y}$ est un *rétracte de voisinage* (non nécessairement compact). $\mathcal{Y}^{\mathcal{X}}$ désigne l'espace des transformations continues de $\mathcal{X}$ en sous-ensembles de $\mathcal{Y}$, en entendant par convergence des fonctions-éléments de $\mathcal{Y}^{\mathcal{X}}$ la *convergence continue*; c'est-à-dire en convenant que $f = \lim f_k$ lorsque la condition $\lim x_k = x$ entraîne $\lim f_k(x_k) = f(x)$.

Par $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ nous désignons *la famille des composantes* (sous-ensembles connexes maximaux) de l'espace $\mathcal{Y}^{\mathcal{X}}$. La notion de convergence des composantes est imposée par celle de convergence continue de fonctions (voir N° 3); $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ devient ainsi un espace *topologique*. Plus encore: cet espace est *métrisable* et — comme nous allons montrer — *homéomorphe à un sous-ensemble fermé de l'espace des nombres irrationnels* (théorème 3); dans le cas particulier où $\mathcal{X}$ est *compact*, l'espace $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ est toujours *discret*.

C'est bien ce théorème et l'étude de l'espace $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ qui constituent le but de cette Note.

1. Généralités sur l'espace $\mathcal{Y}^{\mathcal{X}}$. Comme localement compact, l'espace $\mathcal{X}$ se laisse développer en une série d'ensembles compacts ([6], v. II, p. 51):

1.1. $\mathcal{X} = F_1 \cup F_2 \cup \ldots \cup F_i \cup \ldots,$ *où* $F_i \subset \mathrm{Int}(F_{i+1}).$

---

*) Comme l'a remarqué M. S. Mrówka, plusieurs théorèmes de cette Note se laissent étendre au cas où $\mathcal{X}$ est normal et localement bicompact.

En désignant par $f|F$ la fonction partielle réduite à F, on constate aussitôt que

1.2 *L'opération $f|F$, pour F fixe, est continue* *).

On en déduit facilement l'équivalence suivante

1.3 $(f = \lim\limits_{k=\infty} f_k) \equiv [(f|F_i) = \lim\limits_{k=\infty}(f_k|F_i)$ *quel que soit* $i]$,

qui veut dire que la convergence continue équivaut à la convergence nommée „presque uniforme"; bien entendu, la convergence continue sur un espace compact équivaut à la convergence uniforme.

De là on conclut que l'espace $\mathcal{Y}^{\mathcal{X}}$ est métrisable en posant (cf. [6], v. II, p. 258, et [1], p. 486):

1.4 $$|f-g| = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|(f|F_i)-(g|F_i)|}{1+|(f|F_i)-(g|F_i)|}$$

où, comme d'habitude, $|(f|F)-(g|F)| = \max\limits_{x \in F} |f(x)-g(x)|$.

En d'autres termes: on a l'équivalence

1.5 $(\lim\limits_{k=\infty} |f_k - f| = 0) \equiv (f = \lim\limits_{k=\infty} f_k)$.

C'est une conséquence directe de 1.3 et de l'équivalence évidente:

$$(\lim_{k=\infty} |f_k - f| = 0) \equiv [\lim_{k=\infty} |(f_k|F_i)-(f|F_i)| = 0 \textit{ quel que soit } i].$$

Désignons par $\Phi|F$, où $\Phi \subset \mathcal{Y}^{\mathcal{X}}$ et $F \subset \mathcal{X}$, l'ensemble des fonctions partielles $f|F$ telles que $f \in \Phi$ (voir [6], v. 2, p. 252). C'est donc le sous-ensemble de l'espace $\mathcal{Y}^F$ composé de toutes les fonctions qui admettent une extension sur $\mathcal{X}$ appartenant à Φ.

On a l'équivalence suivante (qui généralise 1.3):

1.6 $(f \in \overline{\Phi}) \equiv [(f|F_i) \in \overline{\Phi|F_i}$ *quel que soit* $i]$.

En vue de 1.2, il s'agit d'établir l'implication de droite à gauche. Or, soient $\varepsilon > 0$ et $f_i \in \Phi$ tels que

$$|(f_i|F_i)-(f|F_i)| < \varepsilon, \qquad \text{d'où} \qquad |f_i - f| < \varepsilon + 2^{-i},$$

d'après 1.4. Donc $f \in \overline{\Phi}$.

1.7 *Φ et Φ^* étant fermés, on a:*

$$(\Phi = \Phi^*) \equiv [\Phi|F_i = \Phi^*|F_i \textit{ quel que soit } i].$$

1.7 résulte de 1.6. Nous déduirons aussi de 1.6 que:

1.8. *L'espace $\mathcal{Y}^{\mathcal{X}}$ est séparable.*

---

*) De plus, l'opération $f|F$ est intérieure, si $\mathcal{Y}$ est compact [7].

Ceci est vrai dans le cas particulier, où $\mathcal{X}$ est compact (cf. [6], v. I, p. 120). L'espace $\mathcal{Y}^{F_i}$ est donc séparable et, par conséquent, $\mathcal{Y}^{\mathcal{X}}|F_i$ qui en est un sous-ensemble, contient un ensemble dense dénombrable. Il existe donc un ensemble dénombrable $\Phi_i \subset \mathcal{Y}^{\mathcal{X}}$ tel, que $\mathcal{Y}^{\mathcal{X}}|F_i \subset \overline{\Phi_i|F_i}$. Posons $\Phi = \Phi_1 \cup \Phi_2 \cup \ldots$ Soit $f \in \mathcal{Y}^{\mathcal{X}}$. On a donc

$$(f|F_i) \in (\mathcal{Y}^{\mathcal{X}}|F_i) \subset \overline{\Phi_i|F_i} \subset \overline{\Phi|F_i}$$

quel que soit i; d'où $f \in \overline{\Phi}$ d'après 1.6.

2. Famille $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ des composantes de l'espace $\mathcal{Y}^{\mathcal{X}}$. Rappelons d'abord que (cf. [3], p. 224 et [6], v. 2, p. 260, 3):

2.1. *Si $\mathcal{X}$ est compact, $\mathcal{Y}^{\mathcal{X}}$ est un rétracte de voisinage; ses composantes sont donc ouvertes et connexes par arcs; en conséquence, si les fonctions f_1 et f_2 appartiennent à la même composante, elles sont homotopes. En outre, si $\lim f_k = f$, tous les f_k, pour k suffisamment grand, appartiennent à la même composante que f.*

2.2. *Si Γ est une composante de $\mathcal{Y}^{\mathcal{X}}$, $\Gamma|F$ en est une de $\mathcal{Y}^F$.*

En effet, l'opération $f|F$ étant continue (cf. 1.2), l'ensemble $\Gamma|F$ est connexe (comme image de l'ensemble connexe Γ). Il existe donc une composante Δ de $\mathcal{Y}^F$ telle que $(\Gamma|F) \subset \Delta$. Il s'agit de montrer que $\Delta \subset (\Gamma|F)$, c'est-à-dire qu'étant donné $g \in \Delta$, il existe une fonction $g^* \in \Gamma$ telle que $g = g^*|F$.

Or, soit $f \in \Gamma$. On a donc $(f|F) \in (\Gamma|F) \subset \Delta$ et d'après 2.1, les fonctions $f|F$ et g sont homotopes. En vertu d'un théorème connu de Borsuk ([2], p. 103, cf. [6], v. 2, p. 278, 3), la fonction g admet donc une extension $g^* \in \mathcal{Y}^{\mathcal{X}}$, homotope à f; cela veut dire (cf. [6], v. 2, p. 276, 6) que g^* se laisse unir à f par un arc $\subset \mathcal{Y}^{\mathcal{X}}$. Donc $g^* \in \Gamma$.

2.3. *Γ étant une composante de $\mathcal{Y}^{\mathcal{X}}$, on a l'équivalence*

$$(f \in \Gamma) \equiv [(f|F_i) \in (\Gamma|F_i) \text{ quel que soit } i].$$

Car Γ et $\Gamma|F_i$, comme composantes, sont fermées.

Il en résulte que (cf. 1.7):

2.4. *Γ et Γ^* étant deux composantes de $\mathcal{Y}^{\mathcal{X}}$, on a:*

$$(\Gamma = \Gamma^*) \equiv (\Gamma|F_i = \Gamma^*|F_i \text{ quel que soit } i).$$

2.5. *Φ étant un sous-ensemble fermé de $\mathcal{Y}^{\mathcal{X}}$, on a l'équivalence*

$$[\Phi \in \mathfrak{L}(\mathcal{Y}^{\mathcal{X}})] \equiv [(\Phi|F_i) \in \mathfrak{L}(\mathcal{Y}^{F_i}) \text{ quel que soit } i].$$

En vue de 2.2, il reste à établir l'implication de droite à gauche. Soient $g \in \Phi$ et $g \in \Gamma \in \mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$. Il s'agit de montrer que $\Phi = \Gamma$. Cette identité résulte de 1.7. Car, pour i fixe, $\Phi|F_i$ et $\Gamma|F_i$ sont des composantes de $\mathcal{Y}^{F_i}$ (la première par hypothèse, et la deuxième — d'après 2.2), non disjointes (puisque $g|F_i$ est leur élément commun), donc identiques.

3. Définition de la convergence des composantes.

Théorème 1. *La décomposition de l'espace $\mathcal{Y}^{\mathfrak{X}}$ en composantes est fermée.* En d'autres termes, *étant données: une suite de composantes $\Gamma, \Gamma_1, \Gamma_2, \ldots$ et deux suites de fonctions $f, f_1, f_2, \ldots$ et $g, g_1, g_2, \ldots$ telles que*

$$(1) \qquad \lim f_k = f, \quad \lim g_k = g, \quad f_k, g_k \in \Gamma_k \quad \text{et} \quad f \in \Gamma,$$

on a $g \in \Gamma$.

Démonstration. Soit i fixe. D'après (1) et 1.3, on a

$$\lim_{k=\infty}(f_k|F_i) = f|F_i \quad \text{et} \quad \lim_{k=\infty}(g_k|F_i) = g|F_i\,.$$

$\Gamma_k|F_i$ et $\Gamma|F_i$ étant des composantes de $\mathcal{Y}^{F_i}$ (d'après 2.2), on conclut de 2.1 que $\Gamma_k|F_i = \Gamma|F_i$ pour k suffisamment grand. Comme $(g_k|F_i) \in (\Gamma_k|F_i)$ d'après (1), on a donc

$$(g_k|F_i) \in (\Gamma|F_i), \quad \text{d'où} \quad (g|F_i) \in (\Gamma|F_i)$$

d'après 2.1 et (1) (deuxième égalité). Il vient $g \in \Gamma$ en raison de 2.3.

Remarque. La décomposition de tout espace compact en composantes est fermée (comme semi-continue supérieurement, cf. [6], v. 2, p. 122). Cependant, il n'en est plus ainsi pour les espaces localement compacts; comme exemple on peut considérer l'espace (situé sur le plan) formé des deux droites horizontales $y = \pm 1$ et de la suite d'ellipses: $x^2 + (n+1)^2 y^2 = n^2$.

Le théorème 1 donne lieu à la définition suivante:

Définition 1. *La composante Γ de l'espace $\mathcal{Y}^{\mathfrak{X}}$ est la limite de la suite de composantes $\Gamma_1, \Gamma_2, \ldots$, en symbole: $\Gamma = \mathrm{Lim}\,\Gamma_k$, lorsqu'il existe une suite $f, f_1, f_2, \ldots$ telle que*

$$(2) \qquad f_k \in \Gamma_k \quad \text{pour} \quad k = 1, 2, \ldots, \quad \text{et} \quad \lim_{k=\infty} f_k = f \in \Gamma\ ^{*}).$$

Soit $\Gamma(f)$ la composante de $f \in \mathcal{Y}^{\mathfrak{X}}$. D'après la définition 1:

3.1. *$\Gamma(f)$ est une transformation continue de l'espace $\mathcal{Y}^{\mathfrak{X}}$ sur l'espace $\mathfrak{L}(\mathcal{Y}^{\mathfrak{X}})$.*

3.2. *Dans le cas où $\mathfrak{X}$ est compact, la condition $\Gamma = \mathrm{Lim}\,\Gamma_k$ implique que $\Gamma_k = \Gamma$ pour k suffisamment grand. Dans ce cas l'espace $\mathfrak{L}(\mathcal{Y}^{\mathfrak{X}})$ est donc discret.*

C'est une conséquence directe de 2.1.

3.3. *F étant un sous-ensemble compact de $\mathfrak{X}$, $\Gamma|F$ est une transformation continue de l'espace $\mathfrak{L}(\mathcal{Y}^{\mathfrak{X}})$ en sous-ensemble de l'espace $\mathfrak{L}(\mathcal{Y}^{F})$.*

---

*) La topologie de $\mathfrak{L}(\mathcal{Y}^{\mathfrak{X}})$, ainsi conçue, est la „topologie-quotient" dans le sens [4], § 9.

Soit, en effet, $\operatorname{Lim}\Gamma_k = \Gamma$. Les conditions (2) étant supposées verifiées, il vient d'après 2.2 et 1.3:

$$(f|F) \in (\Gamma|F) \in \mathfrak{L}(\mathcal{Y}^F), \qquad (f_k|F) \in (\Gamma_k|F) \in \mathfrak{L}(\mathcal{Y}^F) \qquad \text{et} \qquad \lim_{k=\infty}(f_k|F) = f|F\,.$$

Donc $\operatorname{Lim}(\Gamma_k|F) = \Gamma|F$.

Théorème 2. *Soit $\Gamma_1, \Gamma_2, \ldots$ une suite de composantes de l'espace $\mathcal{Y}^{\mathcal{X}}$. S'il existe une suite d'entiers $m_1, m_2, \ldots$ telle que*

$$(3) \qquad \Gamma_k|F_i = \Gamma_{m_i}|F_i \quad \textit{pour} \quad k > m_i\,,$$

la suite $\Gamma_1, \Gamma_2, \ldots$ est convergente.

Démonstration. Il est évidemment légitime d'admettre que $m_1 < m_2 < \ldots$ Nous allons définir une suite $f_{m_1}, f_{m_2}, \ldots$ telle que

$$(4) \qquad f_{m_i} \in \Gamma_{m_i}\,,$$

$$(5) \qquad f_{m_i}(x) = f_{m_{i+1}}(x) \quad \text{pour} \quad x \in F_i\,.$$

Procédons par induction. Soit f_{m_1} une fonction arbitraire satisfaisant à (4) pour $i = 1$. Admettons que la condition (4) est satisfaite pour un i donné ($\geqslant 1$). En posant $k = m_{i+1}$ dans la formule (3) et en la rapprochant de (4), on voit que $(f_{m_i}|F_i) \in (\Gamma_{m_{i+1}}|F_i)$. Il existe donc une fonction $f_{m_{i+1}}$ satisfaisant à (4) pour $i+1$ (c'est-à-dire que $f_{m_{i+1}} \in \Gamma_{m_{i+1}}$) et telle que $f_{m_i}|F_i = f_{m_{i+1}}|F_i$, c'est-à-dire que la condition (5) est réalisée.

La suite $f_{m_1}, f_{m_2}, \ldots$ étant ainsi définie, la condition (5) implique l'existence d'une fonction f telle que $f(x) = f_{m_i}(x)$ pour tout $x \in F_i$ et tout $i = 1, 2, \ldots$ Comme $F_i \subset \operatorname{Int}(F_{i+1})$ (d'après 1.1), la fonction f est continue: $f \in \mathcal{Y}^{\mathcal{X}}$. Soit Γ sa composante. On constate aussitôt (cf. 1.3) que

$$f = \lim_{i=\infty} f_{m_i}\,, \qquad \text{d'où} \qquad \Gamma = \operatorname*{Lim}_{i=\infty} \Gamma_{m_i}\,.$$

Afin de montrer que $\Gamma = \operatorname{Lim}\Gamma_k$, il suffit de définir les fonctions f_k pour $m_i < k < m_{i+1}$ de façon que

$$(6) \qquad f_k \in \Gamma_k \quad \text{et que} \quad f_k(x) = f_{m_i}(x) \quad \text{pour} \quad x \in F_i\,.$$

Or, d'après (3) et (4) on a $(f_{m_i}|F_i) \in (\Gamma_k|F_i)$, ce qui veut dire qu'il existe une fonction f_k satisfaisant à (6).

Corollaire. *On a les équivalences suivantes:*

$$(\Gamma = \operatorname*{Lim}_{k=\infty}\Gamma_k) \equiv [(\Gamma|F_i) = \operatorname*{Lim}_{k=\infty}(\Gamma_k|F_i) \textit{ quel que soit } i] \equiv$$
$$\equiv [\textit{il existe une suite } \{m_i\} \textit{ telle que } \Gamma_k|F_i = \Gamma|F_i \quad \textit{pour} \quad k > m_i]\,.$$

C'est une conséquence de 3.3, 3.2, du théorème 2 et de 2.4.

4. Métrisation de l'espace $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ des composantes de $\mathcal{Y}^{\mathcal{X}}$.

Le théorème 2.4 donne lieu à la définition suivante de la distance $\mathrm{d}(\Gamma, \Gamma^*)$ entre éléments de $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$.

DÉFINITION 2. *Si $\Gamma \neq \Gamma^*$ et si i est le plus petit indice tel que $\Gamma|F_i \neq \Gamma^*|F_i$, nous posons* $\mathrm{d}(\Gamma, \Gamma^*) = 1/i$.

Si $\Gamma|F_i = \Gamma^*|F_i$ pour tout i, nous posons $\mathrm{d}(\Gamma, \Gamma^*) = 0$.

Autrement dit

4.1. $(\mathrm{d}(\Gamma, \Gamma^*) < 1/i) \equiv (\Gamma|F_i = \Gamma^*|F_i)$.

On constate aussitôt que la distance, ainsi définie, satisfait aux axiomes habituellement admis, qui caractérisent les espaces métriques.

Nous allons démontrer qu'elle est compatible avec la notion de convergence définie au N° 3. Autrement dit, que:

4.2. $[\lim\limits_{k=\infty} \mathrm{d}(\Gamma_k, \Gamma) = 0] \equiv (\operatorname*{Lim}\limits_{k=\infty} \Gamma_k = \Gamma)$.

En effet, la condition $\lim \mathrm{d}(\Gamma_k, \Gamma) = 0$ veut dire qu'il existe une suite $\{m_i\}$ telle que $\mathrm{d}(\Gamma_k, \Gamma) < 1/i$ pour $k > m_i$, donc d'après 4.1 que $\Gamma_k|F_i = \Gamma|F_i$, et cela équivaut à $\Gamma = \mathrm{Lim}\,\Gamma_k$, selon le corollaire du N° 3.

THÉORÈME 3. *L'espace $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ est complet, séparable et 0-dimensionnel. Il est donc* ([6], v. 1, p. 348) *homéomorphe à un sous-ensemble fermé de l'espace des nombres irrationnels.*

L'espace est *complet*. Soient, en effet, $\Gamma_1, \Gamma_2, \dots$ une suite de composantes et $m_1, m_2, \dots$ une suite d'entiers telles que, pour $k > m_i$, $\mathrm{d}(\Gamma_k, \Gamma_{m_i}) < 1/i$, c'est-à-dire que (cf. 4.1) la condition (3) est satisfaite. En vertu du théorème 2 la suite $\Gamma_1, \Gamma_2, \dots$ est donc convergente.

L'espace est *séparable*, comme image continue de $\mathcal{Y}^{\mathcal{X}}$, qui est un espace séparable (cf. 1.8 et 3.1).

L'espace est 0-*dimensionnel*, puisque l'ensemble des distances mutuelles de ses éléments est dénombrable.

EXEMPLES ET REMARQUES 1° $\mathcal{X}$ désignant l'espace des entiers positifs et $\mathcal{Y}$ se réduisant à deux éléments, $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ est homéomorphe au discontinu de Cantor.

2° Des nombreux théorèmes de cette Note restent valables dans le cas plus général où l'on omet l'hypothèse de compacité locale de $\mathcal{X}$. On définit alors la topologie dans $\mathcal{Y}^{\mathcal{X}}$ par l'équivalence 1.6 (en y remplaçant F_i par F compact arbitraire $\subset \mathcal{X}$) *) et la topologie dans $\mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ — par l'équivalence:

$$(\Gamma \in \overline{\mathfrak{G}}) \equiv [(\Gamma|F) \in (\mathfrak{G}|F) \textit{ quel que soit } F \textit{ compact}],$$

où $\mathfrak{G} \subset \mathfrak{L}(\mathcal{Y}^{\mathcal{X}})$ (cf. un procédé analogue dans [4], § 7).

---

*) D'après une remarque de R. Engelking, cette topologie est "compact-open" dans le sens de Fox [5] (donc une k-topologie suivant Arens [1]).

3° Dans le cas où $\mathfrak{X}$ est un sous-ensemble de l'espace euclidien à n dimensions et $\mathcal{Y} = \mathcal{S}_{n-1}$ (sphère à $n-1$ dimensions), $\mathfrak{L}(\mathcal{Y}^{\mathfrak{X}})$ est un groupe commutatif relativement à la multiplication cohomotopique de Borsuk. Si $\mathfrak{X}$ est localement compact, ce groupe est isomorphe à un sous-groupe du groupe (additif) des suites infinies de nombres entiers; il est isomorphe à ce groupe tout entier (qui est évidemment homéomorphe à l'espace de tous les nombres irrationnels), si $\mathfrak{X}$ est un ensemble ouvert et $\mathfrak{L}(\mathcal{S}_n - \mathfrak{X})$ est infini.

La démonstration de ces théorèmes paraîtra prochainement.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] R. F. Arens, *A topology for spaces of transformations*, Annals of Math. **47** (1946), 480-495.

[2] K. Borsuk, *Sur les prolongements des transformations continues*, Fund. Math. **28** (1936), 99-110.

[3] — *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), 220-242.

[4] N. Bourbaki, *Topologie Générale*, Chap. I, Paris 1940.

[5] R. H. Fox, *On topologies for function spaces*, Bull. Amer. Math. Soc. **51** (1945), 429-432.

[6] K. Kuratowski, *Topologie*, vol. I-II, Monogr. Mat., Nouv. éd. 1958.

[7] R. Sikorski, *On the operation of cutting of mappings*, Prace Mat. **1** (1955), 136-140.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

MATHEMATICS

On the Characteristic Exponents of the Second Order Linear Ordinary Differential Equation

by

C. OLECH

Presented by T. WAŻEWSKI on July 4, 1958

In this paper we shall give estimates (lower and upper bounds) for the characteristic exponents of solutions *) of the differential equation

(1) $$x'' + a(t)x = 0\,,$$

where the function $a(t)$ is piecewise continuous and stays between two positive bounds, i. e. we suppose that

(2) $$0 < \alpha^2 \leqslant a(t) \leqslant \beta^2 \quad (\alpha, \beta \text{ are constants}).$$

Some upper bounds for characteristic exponents of the solutions of Eq. (1) may be derived from the work of Liapounoff [3]. Other, more accurate, bounds have been obtained by Loud in his recently published paper [4]. Our estimates (see Theorem 1) are the best possible if, beside (2), we know nothing about $a(t)$ (see Theorem 2). We also consider the case of Eq. (1) when $a(t)$ satisfies only the unilateral restriction

(3) $$0 < \alpha^2 \leqslant a(t)\,.$$

In this case Eq. (1) admits solutions possessing unbounded characteristic exponents (see Theorem 3).

Our considerations are related to those of Wintner [5]. We may conclude, judging by our results, that there are some shortcomings in Wintner's paper (see Remarks 3 and 4 of this paper).

1. THEOREM 1. *Let $a(t)$ be real, piecewise continuous, and let $a(t)$ satisfy inequality* (2) *for all $t \geqslant 0$. Then there exist finite constants ϱ_1 and ϱ_2,*

---

*) Number λ_0 is called a characteristic exponent of the function $x(t)$ if $\lambda_0 = \limsup\limits_{t\to+\infty} (1/t)\ln|(x)t|$.

uniquely determined by α and β, such that for the arbitrary, nontrivial solution $x(t)$ of (1) *we have*

$$\limsup_{t\to+\infty} |x(t)|\exp(-\varrho_1 t) > 0 \quad \textit{and} \quad \limsup_{t\to+\infty} |x'(t)|\exp(-\varrho_1 t) > 0\,,$$

and, at the same time, the functions

$$|x(t)|\exp(-\varrho_2 t) \quad \textit{and} \quad |x'(t)|\exp(-\varrho_2 t) \quad \textit{are bounded.}$$

ϱ_1 and ϱ_2 are unbounded if α is fixed and $\beta\to+\infty$.

Remark 1. Let λ_0 be the characteristic exponent of the solution $x(t)$. From Theorem 1 we derive that the function $|x(t)|\exp(-\varrho t)$ is unbounded if $\varrho < \varrho_1$, and converges to zero, if $\varrho > \varrho_2$. But this means that $\varrho_1 \leqslant \lambda_0 \leqslant \varrho_2$.

Proof of Theorem 1. Make the transformation $y = \exp(-\varrho t)x$ in (1). We obtain the equation

$$y'' + 2\varrho y' + \big(\varrho^2 + a(t)\big)y = 0\,. \tag{4}$$

Instead of (4) we will consider the system (5) equivalent to (4)

$$u' = v, \quad v' = -\big(\varrho^2 + a(t)\big)u - 2\varrho v\,. \tag{5}$$

Theorem 1 will have been proved if we show that there exist constants ϱ_1 and ϱ_2 such that

1° for $\varrho = \varrho_1$ system (5) has no nontrivial solution converging to zero if $t\to+\infty$,

2° for $\varrho = \varrho_2$ all solutions of (5) are bounded.

In order to prove 1° let us introduce into our considerations an auxiliary system

$$u' = v,\ v' = \begin{cases} -(\varrho^2+\beta^2)u - 2\varrho v & \text{if} \quad uv \geqslant 0, \quad v \neq 0\,, \\ -(\varrho^2+\alpha^2)u - 2\varrho v & \text{if} \quad uv \leqslant 0, \quad u \neq 0\,. \end{cases} \tag{6}$$

Note that system (6) is autonomous, all its solutions are oscillatory and, if $(u(t), v(t))$ is the solution of (6) $(ku(t), kv(t))$, where k is constant, is also a solution of (6). Thus, we can ascertain, for instance, that, if (6) has one periodic solution, then all nontrivial solutions are also periodic, and they have a common period. Let now $(u(t), v(t))$ be the solution of (6) such that $u(0) = 0$ and $v(0) = 1$, and let $V(\varrho, \alpha, \beta)$ denote the value of $v(t)$ for $t = t_1$, where t_1 is the smallest positive number in which $u(t)$ vanishes. It is easy to demonstrate that if

$$V(\varrho, \alpha, \beta) = -1\,, \tag{7}$$

then all solutions of (6) are periodic and, of course, bounded for $-\infty < t < +\infty$. The system (6) is so simple that we can easily calculate

$V(\varrho, \alpha, \beta)$. If we do so, we obtain

$$V(\varrho, \alpha, \beta) = -\sqrt{\frac{\varrho^2+\alpha^2}{\varrho^2+\beta^2}} \exp\left(-\varrho\left(\frac{1}{\alpha}\operatorname{arc\,ctg}\frac{-\varrho}{\alpha}+\frac{1}{\beta}\operatorname{arc\,ctg}\frac{\varrho}{\beta}\right)\right). \tag{8}$$

If α and β are fixed, function $V(\varrho, \alpha, \beta)$ is strictly increasing for $\varrho < 0$. From the preceding and from the statements:

$$V(0, \alpha, \beta) = -\frac{\alpha}{\beta} \geqslant -1; \quad \lim_{\varrho\to-\infty} V(\varrho, \alpha, \beta) = -\infty,$$

we conclude that (7) has only one non-positive root, which is equal to zero if, and only if, $\alpha = \beta$. Let us denote this root by ϱ_1. Now we are going to show that 1° is valid for $\varrho = \varrho_1$.

In order to do so, let us define the function $W(u, v)$ as follows: $W(0, v) = v$ if $v > 0$ and $W(u(t), v(t)) = \text{const.}$ for every nontrivial solution $(u(t), v(t))$ of (6), for which $\varrho = \varrho_1$. This function is continuous for all u, v, and is of class C^1 if $uv \neq 0$. It also satisfies the following conditions:

$$\left(\frac{\partial W}{\partial u}\right)^2 + \left(\frac{\partial W}{\partial v}\right)^2 > 0 \tag{9}$$

$$\frac{\partial W}{\partial u} v - \frac{\partial W}{\partial v}\big((\varrho^2+\beta^2)u + 2\varrho v\big) = 0 \quad \text{if} \quad uv > 0 \tag{10}$$

$$\frac{\partial W}{\partial u} v - \frac{\partial W}{\partial v}\big((\varrho^2+\alpha^2)u + 2\varrho v\big) = 0 \quad \text{if} \quad uv < 0 . \tag{11}$$

Since the function $\dfrac{\partial W}{\partial v}$ is continuous for $v \neq 0$, it follows from (9), (10) and (11) that

$$\frac{\partial W}{\partial v} v > 0 \quad \text{if} \quad v \neq 0 . \tag{12}$$

Now let us calculate the first derivative of function $w(t) \overset{\text{df}}{=} W(u(t), v(t))$ where $(u(t), v(t))$ is an arbitrary, nontrivial solution of (5). This derivative exists for all $t > 0$ except those in which $u(t)$ or $v(t)$ vanish. But the set of these exceptional values is denumerable. If we use (10) and (11), we obtain for $w'(t)$ the following formulas:

$$w'(t) = \frac{\partial W}{\partial v}\big(\beta^2 - a(t)\big)u(t) \quad \text{if} \quad u(t)v(t) > 0 , \tag{13}$$

$$w'(t) = \frac{\partial W}{\partial v}\big(\alpha^2 - a(t)\big)u(t) \quad \text{if} \quad u(t)v(t) < 0 . \tag{14}$$

From (13), (14) and (12), and from the assumption (2) we find that $w'(t) \geqslant 0$ for all $t > 0$ except those from a denumerable set in which

$w'(t)$ may not exist. On the basis of the last inequality we obtain inequality $w(t) \geqslant C > 0$ which proves 1°.

Statement 2° may be proved quite similarly as 1°. Let us note only that, in this case, instead of (6), we must consider the system

$$(6') \qquad u' = v, \qquad v' = \begin{cases} -(\varrho^2+\alpha^2)u - 2\varrho v & \text{if} \quad uv \geqslant 0, \quad v \neq 0, \\ -(\varrho^2+\beta^2)u - 2\varrho v & \text{if} \quad uv \leqslant 0, \quad u \neq 0. \end{cases}$$

Constant ϱ_2 is determined here as a root of the following equation similar to (7),

$$(7') \qquad V_*(\varrho, \alpha, \beta) = -\sqrt{\frac{\varrho^2+\beta^2}{\varrho^2+\alpha^2}} \exp\left(-\varrho\left(\frac{1}{\alpha}\operatorname{arc\,ctg}\frac{\varrho}{\alpha} + \frac{1}{\beta}\operatorname{arc\,ctg}\frac{-\varrho}{\beta}\right)\right) = -1.$$

The Eq. (7′) has only one positive root, which we denote by ϱ_2. For $\varrho = \varrho_2$ all solutions of (6′) are periodic.

Now we observe that $\lim_{\beta\to+\infty} V(\varrho, \alpha, \beta) = 0$ if $\varrho < 0$ and $\lim_{\beta\to+\infty} V_*(\varrho, \alpha, \beta) = -\infty$ if $\varrho > 0$, where ϱ and α are fixed. From the preceding we conclude that, if α is fixed and $\beta \to +\infty$, then $\varrho_1 \to -\infty$, and $\varrho_2 \to +\infty$. This completes the proof of Theorem 1.

Remark 2. Since we have

$$V(\varrho, \alpha, \beta) = [V_*(-\varrho, \alpha, \beta)]^{-1},$$

it follows that $\varrho_2 = -\varrho_1$.

2. Theorem 2. *There exists a piecewise continuous function $\bar{a}(t)$ for which inequality* (2) *holds, and the equation*

$$(15) \qquad x'' + \bar{a}(t)x = 0$$

admits two linearly independent solutions $x_1(t)$ and $x_2(t)$ such that their characteristic exponents are ϱ_1 and ϱ_2, respectively.

The proof of this Theorem is based on the following lemma:

Lemma 1. *If $x_1(t)$ and $x_2(t)$ are linearly independent solutions of* (15) *($\bar{a}(t)$ is bounded), and if λ_1 is a characteristic exponent of $x_1(t)$ and λ_2 — of $x_2(t)$, then we have $\lambda_1 + \lambda_2 \geqslant 0$.*

Proof of Theorem 2. Let us return to the system (6). Let $(u(t), v(t))$ be the solution of (6) fulfilling the initial conditions $u(0) = 0$ and $v(0) = 1$. Let us denote by t_0 the smallest positive number in which $v(t)$ vanishes and by t_1 the smallest positive number in which $u(t)$ vanishes. Put $\bar{a}(t) = \beta^2$ for $nt_1 \leqslant t < nt_1 + t_0$ and $\bar{a}(t) = \alpha^2$ for $nt_1 + t_0 \leqslant t < (n+1)t_1$ $(n = 0, 1, 2, \ldots)$.

For the function $\bar{a}(t)$ defined above, if $u(t), v(t)$ is a solution of (6), $x(t) = u(t)\exp(\varrho_1 t)$ is a solution of (15).

Ot course, its characteristic exponent is exactly equal to ϱ_1. Let now $x_2(t)$ be another solution of (15) linearly independent from $x_1(t)$, and let λ_2 be its characteristic exponent. From lemma 2 we get $\lambda_2 \geqslant -\varrho_1 = \varrho_2$,

and from Theorem 1 and Remark 2 we obtain $\lambda_2 \leqslant \varrho_2 = -\varrho_1$. Therefore $\lambda_2 = \varrho_2$. This completes the proof of Theorem 2 *).

3. Let us consider the equation

$$y'' + y' + b(t)y = 0 . \tag{16}$$

With reference to (16) we can derive from Theorem 1 and Theorem 2, respectively, the following consequences.

COROLLARY 1. *If $b(t)$ is piecewise continuous and, for all $t \geqslant 0$, satisfies the inequality $\frac{1}{4} + \alpha^2 \leqslant b(t) \leqslant \frac{1}{4} + \beta^2$, and if $\varrho_2 < \frac{1}{2}$, then every solution of* (16) *will approach zero exponentially as $t \to +\infty$* **).

COROLLARY 2. *If we are given α and β for which $\varrho_2 > \frac{1}{2}$, then there exists a piecewise continuous function $\bar{b}(t)$ or even a continuous one* ***) *which, for all $t \geqslant 0$ remains between $\alpha^2 + \frac{1}{4}$ and $\beta^2 + \frac{1}{4}$, and such a function for which* (16) *has unbounded solutions.*

These two corollaries will follow easily from Theorem 1 and Theorem 2, respectively, if we observe that (16) may be carried into (1) by the transformation $x = \exp(\frac{1}{2}t)y$.

Remark 3. Wintner asserted ([5], p. 114 (ii bis)) that, if $b(t)$ is continuous for $t_0 \leqslant t < +\infty$ and remains between positive bounds as $t \to +\infty$, then all solutions of (16) approach zero exponentially. It is easy to show on the basis of Corollary 2 that Wintner's assertion, without any additional assumptions connected with $b(t)$, is not true.

---

*) Here, as well as in the proof of Theorem 3, we define piecewise constant functions $\bar{a}(t)$ and $\bar{\bar{a}}(t)$, in a similar manner as Caccioppoli [2] in order to prove that (1) under assumption (2) might have unbounded solutions.

**) We say that $x(t)$ approach zero exponentially if there exists $\varepsilon > 0$ for which $x(t) = 0\,(\exp - \varepsilon t))$.

***) In this Corollary and in Theorem 2, we assert that there exists a piecewise continuous function $\bar{b}(t)$ (or $\bar{a}(t)$) and just as in the proof of Theorem 2, it may be discontinuous. This is not an essential moment in our considerations. The discontinuous function $\bar{a}(t)$, may be replaced in Theorem 2 by the continuous one. The latter may be deduced from the following theorem of Wintner ([6], p. 428).

THEOREM W. *If $x_1(t)$ and $x_2(t)$ are linearly independent solutions of* (1), *if function $c(t)$ fulfills the following condition*

$$\int_0^\infty |a(t) - c(t)|\,(x_1^2(t) + x_2^2(t))\,dt < +\infty ,$$

and if $z(t)$ is a solution of the equation $z'' + c(t)z = 0$, then

$$z(t) = x_1(t)\,(d_1 + o(1)) + x_2(t)(d_2 + o(1)) ,$$

where d_1 and d_2 are suitable constants. A similar asymptotic formula is true for $z'(t)$.

Wintner proved this theorem for continuous functions $a(t)$ and $c(t)$, but it also remains valid for piecewise continuous functions.

4. Now we shall discuss the case of (1) when $a(t)$ satisfies the unilateral restriction (3). There are some well known results connected with this case, but under additional assumptions. For example, Biernacki [1] has proved that if $a(t)\to+\infty$ when $t\to+\infty$, and if $a'(t)>0$, then all solutions of (1) are bounded.

The following result of a "negative nature" may be obtained in the case of (1) mentioned above.

THEOREM 3. *Suppose that the function $h(t)$ is continuous and positive for $t\geqslant 0$. Then there exists a piecewise continuous function $\overline{\overline{a}}(t)$ (or even a continuous one) which satisfies* (3) *and possesses the following property: there exists solution $x(t)$ of the equation*

$$x''+\overline{\overline{a}}(t)x=0 \tag{17}$$

such that

$$|x(s_n)|>h(s_n) \qquad (n=1,2,\ldots),$$

where s_n is a suitable sequence tending to infinity as $n\to+\infty$.

This theorem may be deduced from the following lemma:

LEMMA 2. *Let H and A be arbitrary positive constants. One may define function $\overline{\overline{a}}(t)$ for $T_0\leqslant t<T_1$, where $T_1>T_0+1$, in such a way that $\overline{\overline{a}}(t)\geqslant A$ for $T_0\leqslant t<T_1$ and that solution $x(t)$ fulfilling the initial conditions $x(T_0)=0$, $x'(T_0)=1$, for some s, $T_0\leqslant s<T_0+1$, satisfies the inequality*

$$|x(s)|>H,$$

and for $t=T_1$ — the conditions $x(T_1)=0$, $x'(T_1)=1$.

Proof of Lemma 2. Let us put

$$\overline{\overline{a}}(t)=a_1^2 \quad \text{for} \quad T_0\leqslant t<T_0+\frac{\pi}{2a_1},$$

$$\text{and for} \quad T_0+\frac{\pi}{2a_1}+\frac{\pi}{a_2}\leqslant t<T_0+\frac{\pi}{a_1}+\frac{\pi}{a_2},$$

and

$$\overline{\overline{a}}(t)=a_2^2 \quad \text{for} \quad T_0+\frac{\pi}{2a_1}<t<T_0+\frac{\pi}{2a_1}+\frac{\pi}{a_2},$$

where a_1 and a_1 satisfy the following conditions:

$$a_1^2>a_2^2>A, \qquad \frac{\pi}{a_1}+\frac{\pi}{a_2}<1 \quad \text{and} \quad \frac{a_1}{a_2}>H,$$

and let us define

$$\overline{\overline{a}}(t)=a(t-if) \quad \text{for} \quad T_0+if\leqslant t<T_0+(i+1)f,$$

where $f=\dfrac{\pi}{a_1}+\dfrac{\pi}{a_2}$, $i=1,2,\ldots,n$ and $nf>1$. Put $T_1=T_0+nf$.

It may be easily seen that $\bar{\bar{a}}(t)$ given above satisfies the thesis of Lemma 2.

Remark 4. From Theorem 3 we find, for instance, if we put in it $h(t) = \exp(t^2)$, that there exists function $\bar{a}(t)$ for which Eq. (18) has solutions with unbounded characteristic exponents. Of course, the same takes place for equations of type (16). Hence, we conclude that, if function $b(t)$ in (16) satisfies the unilateral restriction $b(t) \geqslant b > \frac{1}{4}$, then (16) may have unbounded solutions, while Wintner asserted ([5], p. 117 (ii* bis)) that, in this case, all solutions of (16) approach zero as $t \to +\infty$. Thus, we find that this assertion of Wintner needs some additional assumptions about $b(t)$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] M. Biernacki, *Sur l'équation différentielle $x''+A(t)x = 0$*, Prace Matematyczno-Fizyczne, **40** (1933), 163-171.

[2] R. Caccioppoli, *Una questione di stabilità*, Rendic. Accad. Lincei, **11** (1930), 251-254.

[3] M. A. Liapounoff, *Problème général de la stabilité du mouvement*, Annales de Toulouse, **9** (1907), 203-474.

[4] W. S. Loud, *Some growth theorems for linear ordinary differential equations*, Trans. Americ. Math. Soc., **85** (1957), 257-264.

[5] A. Wintner, *A criterion for homogeneous linear differential equations with damped solutions*, Jour. of Math. and Mech., **6** (1957), 109-117.

[6] — *On linear perturbations*, Quarterly of Appl. Math., **15** (1957), 428-430.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

ASTRONOMY

Subsystems, Populations and Masses of Stars

by

W. IWANOWSKA

Presented on July 5, 1958

It is widely accepted in stellar dynamics to disregard the effects of stellar encounters in view of the very low stellar density common in our Galaxy. The investigations of J. Jeans, K. Schwarzschild and most completely those of S. Chandrasekhar [1] have proved that the time of relaxation in the solar neighbourhood is greater than the age of the Galaxy by several orders of magnitude. Thus, individual stars are regarded as isolated points moving in the regular galactic field, their motions being independent of their masses.

On the other hand, there exists a close correlation between space velocities and masses of stars in the solar neighbourhood approaching strikingly the law of equipartition of energy (cf. [2]). This fact suggests that mutual dynamical interactions between stars did exist at some time or still do exist. "Indeed it appears even probable" states S. Chandrasekhar in his work ([1], p. 82), "that the initial conditions for the present state of motions were provided during those early times under totally different conditions". This means probably much greater stellar density at some earlier epoch of the Galaxy. One should also take into consideration the dynamical interactions between stars and interstellar clouds, which may be more effective by several orders of magnitude than the interactions between the stars themselves in view of the greater masses of clouds. The evidence of some kinds of young stars being formed in close groups also implies mutual interactions between them in their early times. These arguments seem to prove that, in problems of stellar dynamics, the masses of stars deserve more attention than they have attracted so far.

In what follows I wish to emphasize the importance of this factor for one special problem, namely for the relation between stellar populations and subsystems. For the sake of clarity the following definitions have been adopted in these considerations.

Let the concept of stellar population, not unambiguously used by different anthors, be determined by the birthplace of a star. If a galaxy is composed, like ours, of two main concentrations of stars: the central bulge and the ring of spiral arms, we discern two principal populations of stars born in these two regions. The samples accessible to observation consist mainly of stars originated in the solar neighbourhood (population I), and those escaped from the central bulge (population II).

The concept of a subsystem is usually determined by characteristics of distribution and kinematics of a group of stars:

concentration towards the galactic centre,
concentration towards the galactic plane,
group motion,
dispersion of peculiar velocities.

According to the values of these characteristics three kinds of subsystems are discerned: flat, spherical and intermediate, the last kind being the most frequent. It is clear that flat subsystems consist of stars of population I, spherical ones — of those of population II. But what are the intermediate subsystems built of?

According to the views of the present writer they come into existence in three different ways: partly they are real in the sense that they are formed of stars originated in intermediate zones of the Galaxy. Partly they are mixtures of the two principal populations and will be dissected when suitable population criteria are found. An instructive example of this case is exhibited by the long-period variables which were formerly considered as a single typically intermediate subsystem, and have subsequently been dissected into two different subsystems. A third kind of intermediate subsystems, the most frequent one, consists of stars of population I with small masses: here belong all dwarfs later than F, probably the dwarf-cepheids, and, may be, the white dwarfs.

The present position and motion of a star in the Galaxy is a function of its primary position and velocity, of the time during which it was subjected to the action of the general gravitational field of the Galaxy as well as to the encounters with individual stars and interstellar clouds and — in consequence of these encounters — of its mass. Hence, the kinematical and distribution characteristics of a group of stars determining the subsystem depend on the original conditions of these stars determining the population, on their age and on their mass distribution. Schematically this may be written, for a single star, in the following formula

$$x, y, z, \dot{x}, \dot{y}, \dot{z} = f(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, \quad t, \quad m)$$

i. e. kinematical and distribution characteristics (subsystem)	are determined by	the initial conditions (population)	the age and	the mass distribution

for a group of stars.

A thourough analysis of the character of this composite function presents itself as an interesting and promising task of stellar dynamics. Without professing to solve this problem let us consider the effect of one parameter of this function, the mass on the kinematical and distribution characteristics. Massive stars of population I, i. e. those born in spiral arms, leave their native associations with low relative velocities; they form a typically flat subsystem and maintain these properties with time. Stars of small mass formed out of interstellar matter in spiral arms eave their native clouds with greater relative speeds; subjected to the perturbing action of other stars and clouds they acquire greater velocity dispersion, more eccentric orbits and slower centroid rotation. They evolve into intermediate subsystems. The relations are more complicated for stars of population II. What we observe in our neighbourhood is filtered by a selection rule: there is a preference for small and moderate masses to leave the central bulge in eccentric orbits. The very young

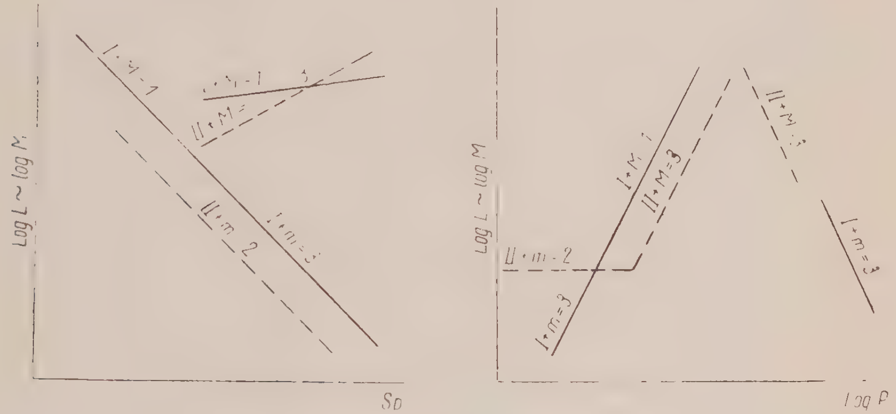

Fig 1

Designation:
I — population I, II — population II;
M — big masses, m — small masses;
1 — flat subsystem, 2 — spherical subsystem,
3 — intermediate subsystem

stars, if there are any in the nucleus of the Galaxy, have not had time to penetrate into our regions. Because of the great relative velocities the dynamical interaction between escaped population II stars and single stars and clouds of spiral arms is not efficient. It seems probable that most of the massive stars of population II observed in our neighbourhood originate from the intermediate zone of the Galaxy and form intermediate subsystems *sensu stricto*. Those of moderate and small mass, like RR Lyrae type stars and subdwarfs, represent spherical subsystems.

Among the kinematical and distribution characteristics determining a subsystem, the concentration towards the galactic centre may be con-

sidered as a primary criterion of population determining the origin of a group of stars. All remaining characteristics, namely the concentration towards the galactic plane, the group velocity and the velocity dispersion are functions of population, age and mass. Thus one could think of a rough method of estimating the average mass and age of a group of stars from these characteristics, its origin being known from the concentration towards the galactic centre, or from some other source of information, e. g. the spectral criteria.

In order to illustrate these considerations we reproduce schematically in Fig. 1 the Hertzsprung-Russell diagram and the period luminosity curve, where the subsystem (1, 2, 3) is determined as a function of population (I, II) and mass (M-big, m-small). For simplification, the age (t) is omitted on these diagrams.

ASTRONOMICAL OBSERVATORY, NICHOLAS COPERNICUS UNIVERSITY, TORUŃ
ASTRONOMICAL INSTITUTE, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] S. Chandrasekhar, *Principles of stellar dynamics*, Chicago, 1942.
[2] A. Opolski, Arkiv för Astr. (1951), 249-255.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

THEORETICAL PHYSICS

On the Order-disorder Transition Temperature in n-Component Alloys

by

S. ŚWIERCZKOWSKI and K. F. WOJCIECHOWSKI

Presented by W. RUBINOWICZ on June 16, 1958

The definition of the order-disorder transition temperature T_t in metallic alloys has not been sufficiently exact till now, even for binary alloys. The aim of this paper is a precise definition of T_t in n-component alloys, arbitrary with respect to crystallic structure and composition. We give also a formula for computing T_t.

The configurational free energy of a n-component alloy may be expressed in the zeroth approximation by long-range order parameters $s_1, s_2, \ldots, s_m$, $0 \leqslant s_i \leqslant 1$, where $m = nl-n-l+1$ and l denotes the number of sublattices in the superlattice of the alloy. We denote by S the system of parameters $\langle s_1, \ldots, s_m \rangle$. Then, as it was shown in [2], we may express the configurational free energy as follows

$$F(S,T) = E(S) - kT \log W(S),$$

where $E(S)$ is the configurational energy, and $k \log W(S)$ is the entropy of the alloy.

It is convenient to define $G(S)$ as equal to $-k \log W(S)$. Then, the above formula for $F(S,T)$ turns into

$$F(S,T) = E(S) + TG(S). \tag{1}$$

As is known, the transition from the state of complete order to the state of complete disorder, which takes place as temperature increases, consists in the transition of the system of atoms in the alloy from a stable state (superlattice) through the metastable (partial disorder) one to the stable state of complete disorder in high temperatures. Let us denote by T_t the transition temperature at which the state of alloy changes from a metastable one to the stable state of complete disorder. If a state defined by a system S corresponding to the temperature T is stable, then

(2) $\min\limits_{S} F(S,T)$ *is attained on the system* S.

Thus, we may define T_t more precisely:

T_t is the lower bound of all temperatures T such that

$$\min_S F(S,T) \text{ is attained only on } S=0 \text{ (i. e. } S=\langle 0,\dots,0\rangle).$$

Let us observe that, from the continuity of $F(S,T)$, it follows that $\min_S F(S,T_t)$ is attained on $S=0$, but not necessarily only on $S=0$. Thus, the long-range order parameters of the system $S^{(t)}=\langle s_1^{(t)},\dots,s_m^{(t)}\rangle$, which corresponds to the transition temperature T_t, may be positive.

In this paper we shall prove that it follows from (1) that the above definition of T_t can be simplified. Namely, we shall show that *T_t is the lowest temperature T satysfying*

$$\min_S F(S,T)=F(0,T). \tag{3}$$

Furthermore, we shall prove that *T_t is given by the following formula*:

$$T_t=\sup_{s>0}\left\{\frac{E(0)-E(S)}{G(S)-G(0)}\right\}, \tag{4}$$

where $E(S)$ and $G(S)$ appear in (1). We shall also verify that (4) gives $T_t\neq 0$. Our definition of T_t is a generalization of a definition proposed in [1] for an AB_3 alloy.

Let us write $\langle s_1,\dots,s_m\rangle=S>0$ if $\max_i s_i>0$. We observe that

$$E(S)<E(0) \quad \text{for} \quad S>0, \tag{5}$$

$$G(S)>G(0) \quad \text{for} \quad S>0. \tag{6}$$

These conditions may be explained as follows. $E(S)$ corresponds to the configurational energy of the alloy; hence, condition (5) asserts that, for $S>0$, a superlattice is possible. Condition (6) follows from the fact that entropy decreases with S.

We shall prove now that if τ is the least temperature satisfying (3) then $\tau=T_t$. It is evident that $T>T_t$ implies (3). Thus, $\tau\leqslant T_t$. It remains to be proved that $\tau<T_t$ is impossible. Let us suppose that $\tau<T_t$, and we shall arrive at a contradiction. Namely, let T' be any temperature satisfying $\tau<T'<T_t$. Then it follows from (1) that for each S

$$F(S,T')-F(S,\tau)=(T'-\tau)G(S). \tag{7}$$

Thus,

$$F(0,T')=F(0,\tau)+(T'-\tau)G(0)$$

and from (6) follows

$$F(0,T')<F(0,\tau)+(T'-\tau)G(S) \tag{8}$$

for every $S>0$.

Now, as it follows from the definition of τ, $\min\limits_{S} F(S, \tau)$ is attained on $S = 0$, and thus (8) implies

$$F(0, T') < F(S, \tau) + (T' - \tau) G(S)$$

for every $S > 0$. Thus, it follows by (7) that $F(0, T') < F(S, T')$ for $S \neq 0$, and this proves that $\min\limits_{S} F(S, T')$ is attained only on $S = 0$. This is in contradiction with $T' < T_t$.

Let us prove (4). It follows from (1) that a temperature T satisfies (3) if, and only if,

$$T \geqslant \frac{E(0) - E(S)}{G(S) - G(0)} \tag{9}$$

for every $S > 0$. Thus, T_t being the lowest of all temperatures T for which (3) holds, is the lowest temperature T satisfying (9), i. e. T_t is given by (4). From (4), (5) and (6) follows $T_t \neq 0$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
DEPARTMENT OF EXPERIMENTAL PHYSICS, WROCŁAW UNIVERSITY

REFERENCES

[1] T. Muto and Y. Takagi, Solid State Phys. 1 (1955), 193.
[2] K. F. Wojciechowski, Acta Phys. Polon. (in press).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

EXPERIMENTAL PHYSICS

The Metastable State of Dye Molecules

by
A. JABŁOŃSKI

Presented on June 23, 1958

The simplest electronic level diagram (Fig. 1) of a luminescent molecule exhibiting three kinds of photoluminescence — fluorescence, phosphorescence and slow fluorescence — was proposed by the present writer many years ago [4], [5]. The metastable level introduced there is considered to be responsible for phosphorescence, as well as for slow fluorescence. Different hypotheses were put forward as to the nature of this state. According to Franck and Livingstone [2] the long lifetime of the state M need not to be due necessarily to electronic selection rules — it can be explained by means of the Franck-Condon prin, ciple by assuming that the atomic nuclei configuration in the state M does not correspond to any configuration in the ground state. Lewis and Kasha [13] assume M to be a triplet state, from which an electronic transition to the singlet ground state N is forbidden by selection rules. Nowadays, the latter hypothesis seems to be generally accepted *). However, neither of the above hypotheses can explain the fact that polarization of the slow fluorescence, arising from $M \rightarrow N$ transition, is either negative **), or, if otherwise,

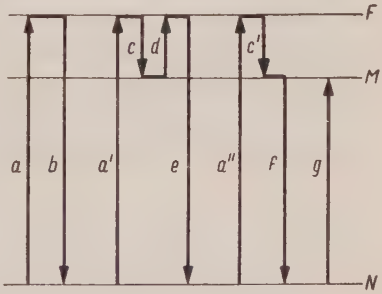

Fig. 1. Electronic energy levels of a phosphorescent molecule. N ground level, M metastable (quasistable) level, F the lowest excited level, which can be reached from N by allowed transition, *ab* fluorescence, *a'cde* phosphorescence, *a''c'f* slow fluorescence, *g* avery weak absorption (observed in some cases)

*) Cf. e. g. a general survey *Spin intercombinations* by M. Kasha and S. P. Mc Glynn [9].

**) The negative polarization of the slow fluorescence was first observed by the present writer while investigating some acridin dyes adsorbed on anisotropic cellophane films, and afterwards also in some isotropic organophosphors [6], [7]. Similar results were obtained by Pringsheim and Vogels for some dyes dissolved in glycerol [18].

at most very low. According to the theory of fundamental polarization [15], [16], [8] polarization of photoluminescence of an isotropic solution can be negative only if the orientation of the transition moment responsible for the emission differs from that responsible for the absorption of the exciting light; this means that the orientation with respect to molecular axes of the $M \to N$ transition moment differs from that of the $F \to N$ transition moment. This fact must be accounted for in any acceptable explanation of the nature of the M state.

The aim of the present paper is to put forward a new hypothesis concerning the nature of the M state.

The luminescent molecules under consideration are plane molecules constituting conjugated systems, in which the familiar absorption and fluorescence spectra may be ascribed to transitions between different states of $2p\pi$ electrons, usually treated as "free" of "metallic" electrons.

The total eigenfunction of a free electron may be written in first approximation as a product of eigenfunctions, one of which is a function of co-ordinate z perpendicular to the plane of the molecule, the other that of x- and y-co-ordinates parallel to the plane of the molecule *). The explicit form of these eigenfunctions is irrelevant to our purpose. For the sake of simplicity, let x represent both the x- and the y-co-ordinates. The total eigenfunction in our approximation is

(1) $$\Psi_{np} = \varphi_n(x)\,\chi_p(z)\,,$$

and the corresponding energy

(2) $$E_{np} = E_n + E_p\,,$$

where n and p are quantum numbers.

The "parallel" transition moment

(3) $$M_x = \int \varphi^*_{n'}(x)\,\chi^*_{p'}(z)\,x\,\varphi_n(x)\,\chi_p(z)\,dx\,dz$$

vanishes, unless $p' = p$, because of the orthogonality of $\chi_p(z)$.

For the same reason the "perpendicular" transition moment

(4) $$M_z = \int \varphi^*_{n'}(x)\,\chi_{p'}(z)\,z\,\varphi_n(x)\,\chi_p(z)\,dx\,dz$$

vanishes, unless $n' = n$.

Thus, the transitions with simultaneous change of both quantum numbers are forbidden in approximation, where $\varphi_n(x)$ is assumed to be absolutely independent of $\chi_p(z)$.

---

*) Often curvilinear co-ordinates along the staggered metallic chain are used for this purpose (cf. e. g. Nikitine and Komoss [14] and Laffitte [10], [11]).

Such forbidden transitions may, however, still occur, although with very small transition probability, if the independence of parallel and perpendicular eigenfunctions is not absolute. They may also occur because of perturbations by molecular vibrations (even by zero point vibrations) and possibly because of some other causes.

It is generally admitted that, in actual cases, the quantum number which we denoted by p remains the same for all levels occupied by free electrons in a molecule in the ground state, as well as for those excited states which have some share in the observed spectra (cf. e. g. Nikitine and Komoss [14]). Let $p = p_1$ for all these levels, and the corresponding energy in Eq. (2) $E_p = E_{p_1}$. We thus have a set of levels: $n = 1, 2, 3, \ldots$; $p = p_1$. For $p \neq p_1$, say $p = p_2$, we obtain a new set of levels ($n = 1, 2, 3, \ldots$; $p = p_2$), which were so far considered to be too high to play any role in the observed spectra. I believe the metastable level M to be the lowest level of this last set: $n = 1$, $p = p_2$. It is metastable because the transition to the level $n = 1$, $p = p_1$ is prevented by the Pauli principle, the last level being fully occupied in actual cases. The only possible transitions are those in which both quantum numbers change, i. e. forbidden transitions, which, as shown above, can occur only with very small transition probability. Their transition moment must have at least a non-vanishing component along the z-axis. Hence, the corresponding fluorescence band (slow fluorescence $M \to N$ band) must be negatively, or, if otherwise, at most very lowly, polarized.

It seems very plausible that, while $p = p_1$ corresponds to the $2p\pi$ electrons, $p = p_2$ corresponds to the state, in which one of the free electrons is a $3s\sigma$ electron. The energy difference between $1s^2 2s^2 2p^2$ and $1s^2 2s^2 2p 3s$ state of the C-atom is 7.45 eV. We may expect roughly the difference of the same order to exist between energies of a free electron in $2p\pi$ and $3s\sigma$ state (n unchanged), and thus estimate the positions of energy levels of molecules, in which one of the free electrons is a $3s\sigma$ electron, the remaining being $2p\pi$ electrons. For the present, we shall restrict our discussion to fluorescein and trypaflavine molecules only. The familiar energy levels (for $p = p_1$, according to our notation) were calculated by Laffitte [10], [11] on the ground of the metallic model *), these levels being given by

$$E_n = \frac{h^2 n^2}{8 m L^2}, \tag{3}$$

where L is the length of the metallic chain.

---

*) Although the ramifications of metallic chains were not taken into account in these calculations, the calculated energy levels agree sufficiently well with the observed spectra, and thus may be used for our purpose.

According to our hypothesis there are two sets of levels:

$$E_{np_1} = \frac{h^2 n^2}{8mL^2} + E_{p_1} \tag{4}$$

(all free electrons $2p\pi$ electrons) and

$$E_{np_2} = \frac{h^2 n^2}{8mL^2} + E_{p_2} \tag{5}$$

(all free electrons but one $2p\pi$, and one $3s\sigma$ electron), where $E_{p_2} - E_{p_1} \approx \approx 7.45$ eV.

In a fluorescein molecule there are 10 free electrons, which occupy all the levels up to $n = 5$ in the ground state N of this molecule. In the F-state one of the $n = 5$ electrons is excited to the $n = 6$ level. The energy levels calculated by means of Eqs. (4) and (5) are: $E_{5p_1} = 5.78$ eV. (N-state), $E_{6p_1} = 8.3$ eV (F-state), $E_{1p_2} = 7.68$ eV (M-state), $E_{2p_2} = 8.37$eV. According to this estimate, there is one level belonging to the p_2 set between F- and N-level 0.6 eV below the F-level (the experimental value quoted by Pringsheim ([17], p. 443) is 0.4 eV).

For a trypaflavine molecule (with 12 free electrons) the calculated energy levels are: $E_{6p_1} = 7.44$ eV (N-state), $E_{7p_1} = 10.11$ eV (F-state), $E_{1p_2} = 7.66$ eV (M-state), $E_{2p_2} = 8.27$ eV, $E_{3p_2} = 9.79$ eV. Our estimate thus leads to the conclusion that there are possibly three levels of the p_2 set in the trypaflavine molecule between F – and N – level, of which the lowest must necessarily be metastable *).

The above estimates are very rough indeed, and are intended only to show the plausibility of our hypothesis on the nature of the M-state.

It should be noted that, while in the p_1 set of levels the angular momenta of all free electrons cancel each other, they do not do so in the p_2 set, whence results the appearance of paramagnetism in molecules excited to the M-state (cf. [12] and [1]).

PHYSICS DEPARTMENT, NICHOLAS COPERNICUS UNIVERSITY, TORUŃ
PHOTOLUMINESCENCE LABORATORY (TORUŃ), INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] D. F. Evans, Nature **176** (1955), 777.
[2] J. Franck and R. Livingstone, J. Chem. Phys. **9** (1941), 184.
[3] J. Franck and P. Pringsheim, J. Chem. Phys. **11** (1943), 21.

*) According to Pringsheim and Vogels [18] and to Franck and Pringsheim [3] there are in trypaflavine two metastable levels. Investigations are carried out in this laboratory with a view to finding out, whether these levels belong to the same or to different isomeric forms of this molecule.

[4] A. Jabłoński, Nature **131** (1933), 839.
[5] — Zs. f. Phys. **94** (1935), 38.
[6] — Bull. intern. Cracow (1935), 30.
[7] — Acta Phys. Polon. **4** (1935), 311.
[8] — Zs. f. Phys. **96** (1935), 236.
[9] M. Kasha and S. P. McGlynn, Ann. Rev. Phys. Chem. **7** (1956), 403.
[10] E. Laffitte, Thèse, Paris (1954).
[11] — Ann. de phys. **10** (1955), 71.
[12] G. N. Lewis, M. Calvin and M. Kasha, J. Chem. Phys. **17** (1949), 804.
[13] G. N. Lewis and M. Kasha, J. Americ. Chem. Soc. **66** (1944), 2100.
[14] S. Nikitine and S. G. Komoss, J. Chim. Phys. **47** (1950), 798.
[15] F. Perrin, Ann. de phys. **12** (1929), 169.
[16] — Acta Phys. Polon. **5** (1936), 335.
[17] P. Pringsheim, *Fluorescence and Phosphorescence,* Interscience Publishers, Inc., New York (1949).
[18] P. Pringsheim and H. Vogels, Acta Phys. Polon. **4** (1935), 341.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

EXPERIMENTAL PHYSICS

Electroluminescence of CdS-Ag with an Ag_2S Layer

by

H. ŁOŻYKOWSKI and H. MĘCZYŃSKA

Presented by A. JABŁOŃSKI on July 4, 1958

In the present investigation an attempt was made to obtain an increase in electroluminescence brightness of a CdS-Ag phosphor by coating it with a layer of Ag_2S, which is a good semiconductor.

For the phosphor thus prepared, the dependence of the electroluminescence on the tension, the frequency and the thickness of the Ag_2S layer was investigated.

Method of preparing CdS-Ag coated with ab Ag_2S layer

The phosphor was coated with an Ag_2S layer by a method used by P. Zalm, G. Diemer and A. Klasens [1] for covering ZnS-Cu with a thin layer of Cu_2S. The CdS-Ag, which has a yellow colouring, was washed in an argentic nitrate solution ($AgNO_3$) of known concentration. The CdS-Ag crystals were then covered with a thin layer of Ag_2S, the phosphor assuming a dark grey colouring depending on the thickness of the Ag_2S layer.

Samples having various thicknesses of the Ag_2S layer were prepared by varying the time of washing in the $AgNO_3$ solution. The quantity of Ag_2S in the sample was determined from the difference between the Ag contents in the $AgNO_3$ solution previous to rinsing the phosphor and that in the filtrate as shown by volumetric analysis.

The electroluminescence of the CdS-Ag phosphor coated with a layer of Ag_2S, as obtained by the above method, was investigated by placing the phosphor within an electroluminescent cell, as shown schematically in Fig. 1.

As a dielectric castor oil was used. One of the electrodes was of conducting glass, the other being of brass.

The electroluminophor was excited with a generator of audiofrequency sinusoidal vibrations ranging from 150 Hz to $5k$Hz.

The tension applied to the electroluminescent cell was measured with a valve voltmeter. The brightness of electroluminescence was observed using a system consisting of an RCA 931A photomultiplier, a valve millivoltmeter and a double beam oscillograph.

The block diagram of the device is shown in Fig. 2.

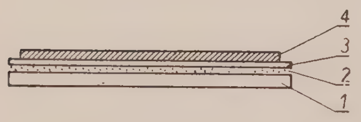

Fig. 1

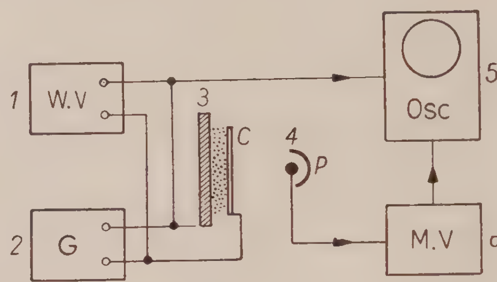

Fig. 2

Fig. 1. Schematic diagram of electroluminescent cell; 1. brass electrode, 2. phosphor in oil, 3. mica, 4. electrode of semiconducting glass

Fig. 2. Block diagram of device; 1. Valve voltmeter, 2. Generator, 3. Electroluminescent cell, 4. Photomultiplier, 5. Double beam oscillograph, 6. Valve millivoltmeter

Results

Investigations of the brightness of electroluminescence of an uncoated CdS-Ag phosphor and that of a CdS-Ag phosphor coated with a layer of Ag_2S showed that the presence of the Ag_2S layer increases the intensity of the electroluminescence.

Moreover, the brightness of electroluminescence was found to increase with the thickness of the Ag_2S layer.

Of all the samples prepared and investigated, that with the largest amount of Ag_2S, equalling 11.62×10^{-2} mg/mg CdS-Ag, yielded an electroluminescence brightness twenty times that of CdS-Ag not coated with Ag_2S, for the same values of the frequency and of the applied tension.

Figs. 3 and 4 show the curves of the brightness of electroluminescence versus the tension applied for CdS-Ag coated with Ag_2S, whereas Fig. 5 shows the corresponding curves for a sample not containing Ag_2S.

For CdS-Ag coated with a layer of Ag_2S, graphs of the natural logarithm of the brightness (B) as function of $10/\sqrt{V}$ (V the applied tension), as well as of $100/V$, are given. The aim of this double representation was to check the data obtained against the theoretical formulae (1) [3] and (2) [2]:

(1) $$B = A\exp\left(-\frac{b}{\sqrt{V}}\right),$$

(2) $$B = A'\exp\left(-\frac{b'}{V}\right).$$

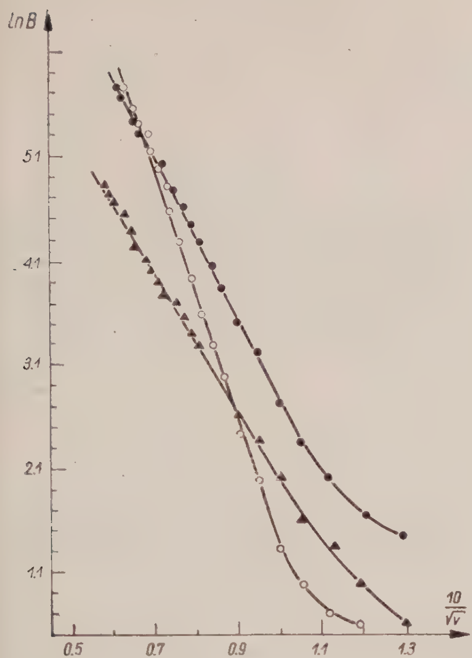

Fig. 3

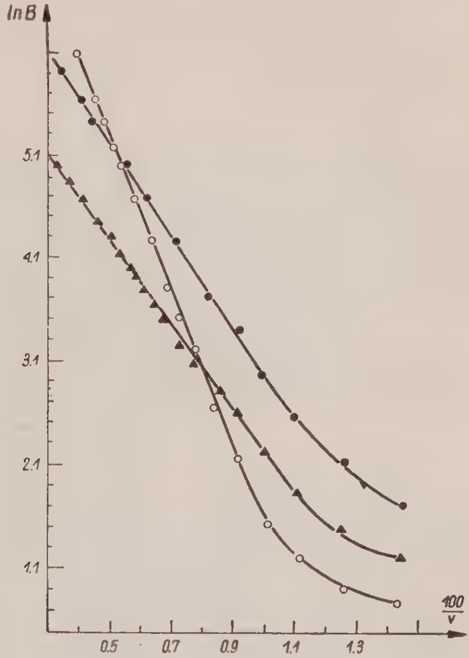

Fig. 4

Fig. 3. Tension dependence of electroluminescence brightness of CdS - Ag coated with Ag_2S; ▲ — 150 Hz • — 500 Hz, ○ — 1000 Hz

Fig. 4. Tension dependence of electroluminescence brightness of CdS - Ag coated with Ag_2S; ▲ —150 Hz, • —500 Hz, — 1000 Hz

Fig. 5

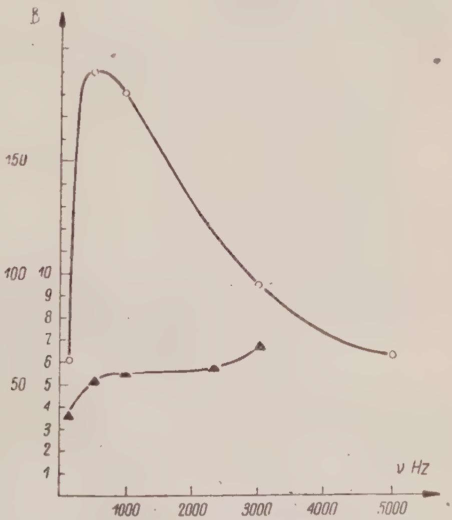

Fig. 6

Fig. 5. Tension dependence of electroluminescence brightness of CdS - Ag (uncoated); ▲ —150 Hz, • —500 Hz, ○ — 500 Hz

Fig. 6. Frequency dependence of electroluminescence brightness ○ — for CdS - Ag coated with Ag_2S; ▲ — for CdS - Ag not coated with Ag_2S

The graphs show some deviation from both (2) and (1). Thus, neither of these formulae describes correctly the behaviour of the luminophor investigated, neither of the curves in Fig. 3 or Fig. 4 being a straight line.

It seems, however, that Eq. (1) agrees better with the experimental results.

The graphs of frequency dependence of the brightness of electroluminescence are shown in Fig. 6 for CdS-Ag coated and not coated with Ag_2S. The brightness of electroluminescence was found to increase with the frequency of the field applied, attaining its maximum at a frequency of 500 Hz for a phosphor coated with a layer of Ag_2S. In the case of a CdS-Ag phosphor not covered with Ag_2S, a different form of the dependence is obtained.

Conclusions from the experimental data

The above results seem to indicate that an important role is played in the electroluminescence by the Ag_2S layer, which constitutes a good semiconductor covering CdS-Ag.

It seems probable that, as a result of the difference in the work of separation from CdS-Ag and from Ag_2S, the electrons pass from CdS-Ag to Ag_2S. Then, Ag_2S assumes the role of donor of electrons.

The electroluminophor samples were also excited optically, those coated with Ag_2S exhibiting a shorter duration of emission as compared to CdS-Ag not covered with Ag_2S.

Further investigations are being carried out with the aim of elucidating the effect observed.

The authors wish to express their very true indebtedness to Professor A. Jabłoński and Dr. A. Wrzesińska for the interest they took in the present investigation and for their valuable discussions.

PHYSICS DEPARTMENT, NICHOLAS COPERNICUS UNIVERSITY, TORUŃ

REFERENCES

[1] P. Zalm, G. Diemer and A. Klasens, Phys. Res. Rep. **9** (1954), 81.
[2] D. Curie, J. Phys. et Radium **14** (1953), 510.
[3] P. Zalm, G. Diemer, A. Klasens, Phys. Res. Rep. **10** (1955), 205.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 9, 1958

EXPERIMENTAL PHYSICS

Non-linear Effects in Rochelle Salt

by

J. STANKOWSKI

Presented by A. SOŁTAN on July 7, 1958

It was the aim of the present paper to investigate the polarization of Rochelle salt below and above the upper Curie point, with the intention of clearing some misunderstandings which appeared recently [3].

The dependence of the total polarization of Rochelle salt on the electric field strength was investigated within the range of 17-35°C by dielectric hysteresis. The total polarization was found to be a non-linear function of the field strength, for strong fields also, and at temperatures exceeding the Curie point.

The Rochelle salt crystals had been grown by the dynamic method. During the process of crystallization, which lasted 10 days on an average, the temperature was lowered from 35° to 33°C. A plate was cut from the crystal perpendicular to a-axis, and the faces were covered with colloidal graphite electrodes.

The usual Sawyer-Tower circuit [1] was used for measurements. The typical shape of the $P(E)$ function below and above the upper Curie point is shown in Fig. 1a, b.

The dependence of the polarization P on the electric field strength amplitude E was obtained from oscillograms of the hysteresis loops (Fig. 2). The curve for 17°C contains points corresponding to high values of E, for which saturation of domain orientation has already been attained, i. e. above the point at which both branches of the hysteresis loop meet. One can see that, on exceeding the Curie point (24°C), the shape of the $P(E)$ dependence changes (Fig. 3). As the temperature rises, the field strength remaining constant ($E = 14.2$ kV/cm), the dependence approaches linearity. It may be shown that the $P(E)$ dependence presented in Fig. 2 is of the type $P^2 = aE$ at 17°C, and of the type $P^{3/2} = bE$ above (27°C and 35°C) the Curie point (Fig. 4). It was not possible to proceed with measurements beyond 35°C because of dielectric breakdown of the samples.

Since the $P(E)$ dependence is non-linear (Fig. 1a) in strong electric fields even after alignment of the domains has been attained, the spontaneous polarization may not be determined by the usual method consisting in tracing the tangent. It seems that this tangent should be traced not too far from the loop (line 2 in Fig. 1a), but at the point at

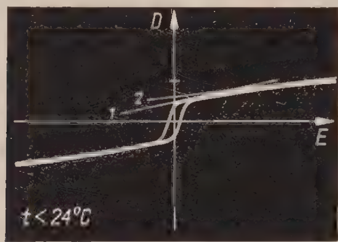

Fig. 1a

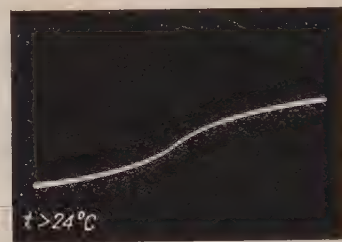

Fig. 1b

Fig. 1. Oscillograms representing the $P(E)$ dependence in Rochelle salt below (a) and above (b) the Curie point

which both branches of the hysteresis loop meet (line 1 in Fig. 1a). This point is determined by complete domain orientational polarization. In fields of still greater strength, induced electronic and ionic polarization takes place only. Since the fields used in the present investigation are unable to produce saturation of the electronic polarization, the non-linearity of this part of the $P(E)$ function must be accounted for by the polarization connected with displacement of the ions.

The spontaneous polarization as determined by the method described, i. e. by tracing the tangent at the origin of the 17°C-curve in Fig. 2,

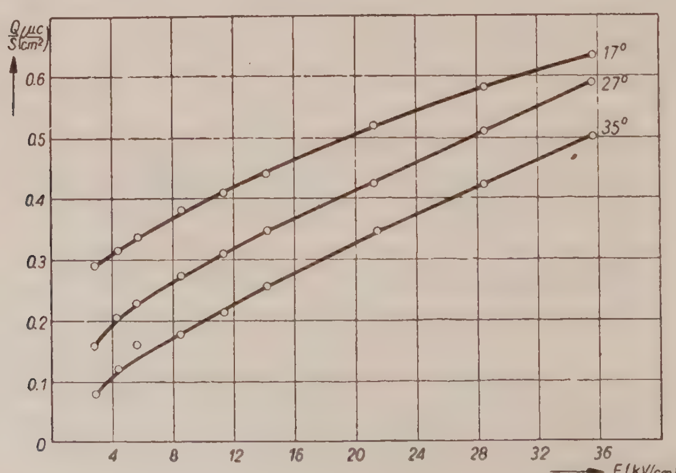

Fig. 2. The $P(E)$ dependence for 17°, 27° and 35°C

amounted to 0.25 $\mu C/\mathrm{cm}^2$. The value is in good agreement with that found in the literature 0.265 $\mu C/\mathrm{cm}^2$ [2].

The results of the present investigation contradict those of Kosman [3], who obtained the dielectric hysteresis loops in Rochelle salt far above 24°C.

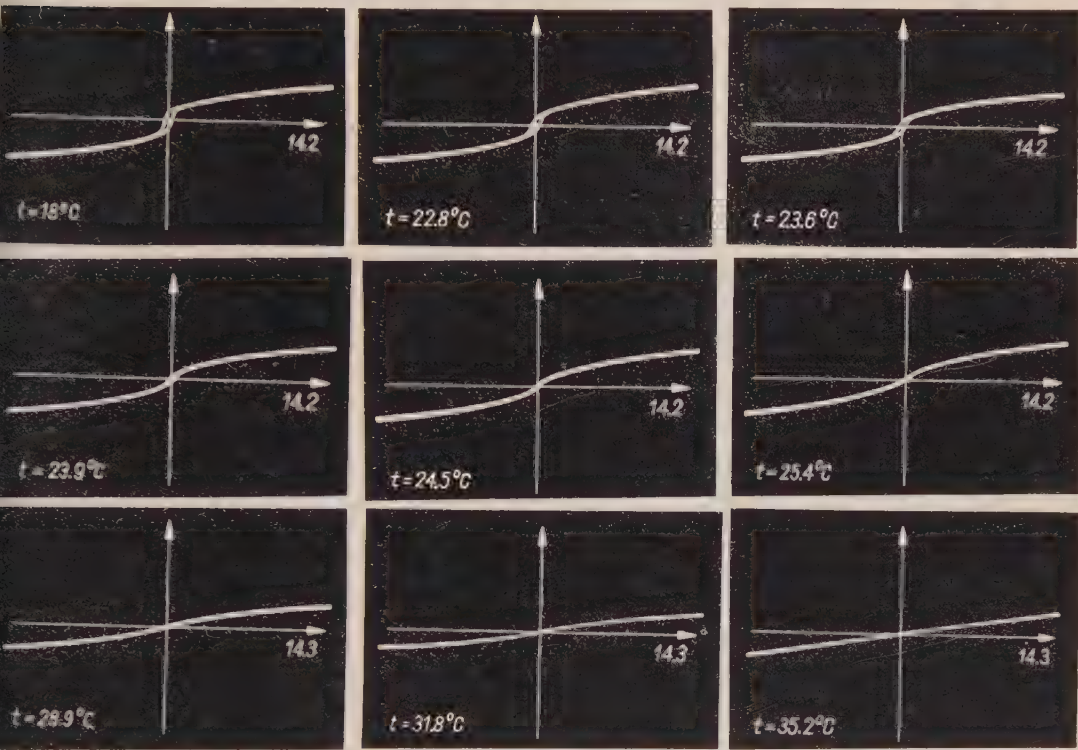

Fig. 3. Oscillograms showing the temperature variation of the $P(E)$ dependence within the range of 18° to 35.2°C, the field being 14.2 kV/cm or 14.3 kV/cm

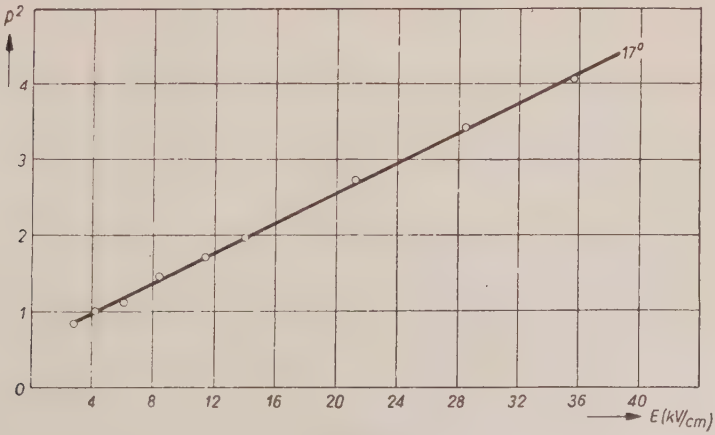

Fig. 4a

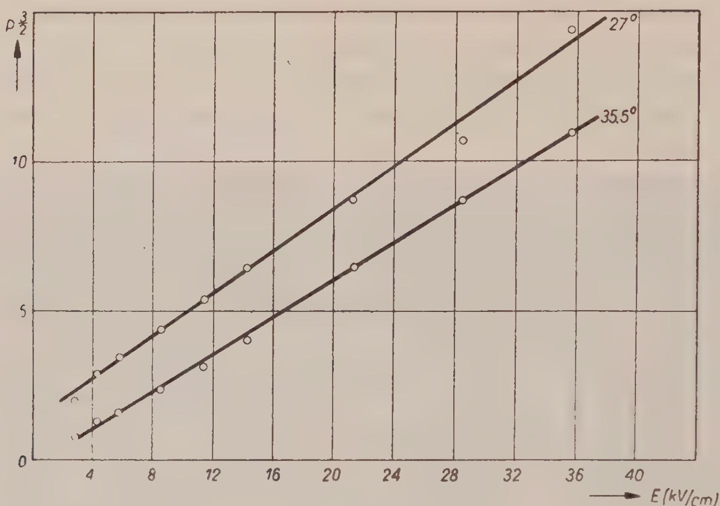

Fig. 4b

Fig. 4. The $P(E)$ dependence below (a) and above (b) the Curie point

However, the measuring circuit used by Kosman, as well as the material investigated, seem to be such as to make his results questionable.

The author wishes to express his indebtedness to Professor A. Piekara for making the investigation possible and for numerous helpful discussions.

DIELECTRIC LABORATORY (POZNAŃ), INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] C. B. Sawyer, C. H. Tower, Phys. Rev., **35** (1930), 269.
[2] H. Mueller, Ann. New York Acad. Sc., **40** (1940), 321.
[3] M. S. Kosman, A. N. Shevardin, Zh. Tc. Phys., **26** (1956), 1443.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 9

С. ЯНКОВСКАЯ, РЕШЕНИЯ СИСТЕМЫ УРАВНЕНИЙ $\varphi(x)=\varphi(y)$ И $\sigma(x)=\sigma(y)$ ДЛЯ $x<y<10.000$ . . . . стр. 541—544

Даны все 86 решений системы уравнений

$$\begin{cases}\varphi(x)=\varphi(y)\\ \sigma(x)=\sigma(y)\end{cases}\tag{1}$$

в натуральных числах $x<y<10000$. Самые малые числа, являющиеся решением вышеприведенной системы, это $x=175$, $y=183$. Среди найденных решений имеется только 5, для которых числа x и y взаимно простые.

Примитивным решением системы уравнений (1) называется такое решение x, y, для которого не существует целое число $a>1$ такое, что $a|x$, $a|y$ и $\left(a, \frac{xy}{a^2}\right)=1$. Имеется только 27 примитивных решений системы уравнений (1), где $x<y<10000$.

Автор ставит вопрос, существует-ли бесконечное множество примитивных решений системы уравнений (1). Имеется бесконечное множество систем натуральных чисел x, y, где $x<y$ и

$$\varphi(x)=\varphi(y), \quad \sigma(x)=\sigma(y), \quad \theta(x)=\theta(y),$$

например: $x_k=3^k\cdot 568$, $y_k=3^k\cdot 638$ для $k=0,1,2,\ldots$

Поставлен вопрос, существует-ли для всякого натурального числа m, m различных натуральных чисел $x_1, x_2, \ldots, x_m$, для которых

$$\varphi(x_1)=\varphi(x_2)=\ldots=\varphi(x_m), \ \ \sigma(x_1)=\sigma(x_2)=\ldots=\sigma(x_m)$$

и

$$\theta(x_1)=\theta(x_2)=\ldots=\theta(x_m).$$

В настоящей работе $\varphi(n)$ обозначает функцию Эйлера-Гаусса, $\sigma(n)$ — сумму натуральных делителей числа n, $\theta(n)$ — их число.

П. ЭРДЕШ, РЕШЕНИЕ ДВУХ ПРОБЛЕМ С. ЯНКОВСКОЙ...стр. 545—548

Решая поставленные С. Янковской в предыдущем сообщении две проблемы, автор доказывает, что:

I. существует бесконечное множество пар натуральных чисел a и b, удовлетворяющих условиям $(a, b) = 1$, $\varphi(a) = \varphi(b)$, $\sigma(a) = \sigma(b)$, $d(a) = d(b)$, где $\varphi(n)$ — функция Эйлера, $\sigma(n)$ — сумма делителей числа n и $d(n)$ — число делителей n;

II. существует для каждого k последовательность k различных натуральных чисел $a_1, a_2, \ldots, a_k$, удовлетворяющих условиям $\varphi(a_i) = \varphi(a_j)$, $\sigma(a_i) = \sigma(a_j)$ и $d(a_i) = d(a_j)$ для $1 \leqslant i < j \leqslant k$. Решение основано на леммах, доказанных автором в его работе [1]. В конце сообщения поставлены новые проблемы, касающиеся функций $\varphi(n)$, $\sigma(n)$ и $d(n)$.

Пусть Φ — непрерывный метод суммируемости, соответствующий последовательности функций $\varphi_n(t)$, непрерывных на $\langle 0, T \rangle$. В работе приводятся условия необходимые и достаточные для того, чтобы при помощи сохраняющего сходимость метода Φ, не суммировались ограниченные и расходящиеся последовательности. Получаются обобщения некоторых теорем Мазура и Орлича, а также Вилянского и Целлера.

В работе доказывается, что если $w = f(z)$ отображает единичный круг $|z| < 1$ на риманову поверхность W так, что $f(0) = 0$, $f'(0) > 0$ и если риманова поверхность W^* полученная из W путем круговой симметризации по отношению к действительной оси — гиперболического типа, тогда функция $f^*(z)$, отображающая $|z| < 1$ на W^* так, что $f^*(0) = 0$, $f^{*\prime}(0) > 0$, быстрее достигает значения с данным модулем, чем $f(z)$.

Отсюда вытекает, что $M(r, f^*) \geqslant M(r, f)$, где $M(r, f) = \sup\limits_{|z|<r} |f(z)|$.

В работе дается краткое доказательство известной теоремы Соболева-Кондрашева (если Ω_N — N-мерная ограниченная область, обладающая сильным коническим свойством, $l \geqslant [N/p] + 1$, $p > 1$, тогда пространство Соболева на Ω_N, $W_p^{(l)}(\Omega_N) \subset C(\Omega_N)$ и вложение $W_p^{(l)}(\Omega_N)$ в $C(\Omega_N)$ является вполне непрерывным). Доказательство опирается на неравенствах Ниренберга [1].

Доказывается, что пространство этих компонент гомеоморфно замкнутому подмножеству пространства всех иррациональных чисел.

В работе приводятся оценки для характеристических показателей решений дифференциального уравнения

$$x'' + a(t)x = 0\,, \tag{1}$$

причем функция $a(t)$ удовлетворяет следующему неравенству

$$0 < \alpha^2 \leqslant a(t) \leqslant \beta^2\,.$$

Доказывается, что полученные оценки наилучшие среди возможных. Изучается также случай уравнения (1), когда $a(t) \geqslant \alpha^2 > 0$.

Полученные результаты опровергают некоторые утверждения работы А. Винтнера. [5].

Автор рассматривает связь между пространственными и кинематическими характеристиками, определяющими подсистему звезд и их происхождением (тип населения), учитывая влияние масс звезд на эти характеристики.

Приводится точное определение температуры перехода порядок-беспорядок в многокомпонентных сплавах. Из этого определения следует, что температуру перехода T_t дает формула

$$T_t = \sup_{S>0} \frac{E(0) - E(S)}{G(S) - G(0)}\,,$$

где $E(S)$ — конфигурационная энергия сплава, а $-G(S)$ — энтропия соответствующая состоянию S.

Предложена новая гипотеза о природе метастабильного уровня M в молекулах красителей. Согласно этой гипотезе в состоянии M один из „металлических" электронов является $3s\sigma$ электроном, все остальные — $2p\pi$ электронами, в отличие от состояний N и F, в которых все металлические электроны являются $2p\pi$ электронами. Дается причина метастабильности уровня M.

В настоящей работе авторы исследовали электролюминесценцию фосфора CdS-Ag, покрытого слоем полупроводника Ag_2S.

Измерена зависимость яркости электролюминесценции от приложенного напряжения, частоты, а также толщины слоя Ag_2S.

Полученные результаты кажутся подтверждать предположение, что важную роль в явлении электролюминесценции играет введение в фосфор полупроводника в виде отдельной фазы.

Я. СТАНКОВСКИЙ, НЕЛИНЕЙНЫЕ ЭФФЕКТЫ В СЕГНЕТОВОЙ СОЛИ . . . . стр. 599—602

Исследована зависимость полной поляризации от электрического поля для сегнетовой соли в пределах температур от 17°-35° С.

Найдено, что функция $P(E)$ — ниже точки Кюри (24° С) в области однозначности функции — типа $P^2 = aE$, тогда как выше точки Кюри функция $P(E)$ имеет вид $P^{3/2} = bE$. Спонтанная поляризация определена при помощи тангенса в точке встречи обеих ветвей петли гистерезиса. Значение спонтанной поляризации, определенной таким образом, равно $0{,}25\ \mu C/см^2$, что хорошо согласуется со значениями данными в литературе.

Приведенные в работе результаты делают сомнительными результаты М. С. Космана и А. Н. Шевардина, согласно которым сегнетоэлектрическое состояние возможно легко вызвать выше точки Кюри. Автор допускает, что причиной разногласия между вышеупомянутыми результатами и результатами полученными в настоящей работе является применение Косманом несоответственной измерительной системы и, может быть, разниц в методике изготовления образцов.

TABLE DES MATIÈRES

Cena zł 20.—